PROBLEM 1 :

(a). YES. $E\bar{X}=\mu=\lambda$, $ES^2=\sigma^2=\lambda$

POISSON HAS MEAN λ AND VARIANCE λ

(b). $p(x)=\dfrac{\lambda^x e^{-\lambda}}{x!}$ $\quad \ln p(x) = x\ln\lambda-\lambda-\ln x!$

$\dfrac{d\ln p(x)}{d\lambda}=\dfrac{x}{\lambda}-1$ $\quad \dfrac{d^2\ln p(x)}{d\lambda^2}=-\dfrac{x}{\lambda^2}$

CRAMER-RAO LOWER BOUND: $\dfrac{1}{-nE\left(-\frac{x}{\lambda^2}\right)}=\dfrac{\lambda}{n}$.

ALSO $VAR(\bar{X})=\frac{\sigma^2}{n}=\dfrac{\lambda}{n}$. THEREFORE $\bar{X}$ IS UMVUE

WHICH MEANS $VAR(\bar{X})\le VAR(S^2)$

(c). $M_{\bar{X}}(t)=\left(M_{X_i}\left(\frac{t}{n}\right)\right)^n=\left\{\left(1-\frac{\beta t}{n}\right)^{-\alpha}\right\}^n=\left(1-\frac{\beta}{n}t\right)^{-n\alpha}$

$\therefore \bar{X}\sim\Gamma\left(n\alpha,\frac{\beta}{n}\right)$

(d). $E\,(n-2)\dfrac{Q_1}{Q_2}=(n-2)(EQ_1)(EQ_2^{-1})$

$Q_2\sim\chi^2_{n-2}$ OR $Q_2\sim\Gamma\left(\frac{n-2}{2},2\right)$

$EQ_2^{-1}=\dfrac{\beta^k\Gamma(\alpha+k)}{\Gamma(\alpha)}=\dfrac{2^{-1}\Gamma\left(\frac{n-2}{2}-1\right)}{\Gamma\left(\frac{n-2}{2}\right)}=\dfrac{1}{2}\dfrac{\Gamma\left(\frac{n-2}{2}-1\right)}{\left(\frac{n-2}{2}-1\right)\Gamma\left(\frac{n-2}{2}-1\right)}$

$=\dfrac{1}{2}\dfrac{1}{\frac{n-4}{2}}=\dfrac{1}{n-4}\rightarrow E(n-2)\dfrac{Q_1}{Q_2}=\dfrac{n-2}{n-4}$.

PROBLEM 2:

(a). $S^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$ WE WANT $E\,cS = \theta$ $\rightarrow$ $c = \frac{\theta}{ES}$

TO FIND ES USE $E(S^2)^{1/2}$

$$ES = E(S^2)^{1/2} = \frac{\left(\frac{2\theta^2}{n-1}\right)^{1/2}\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$= \theta\sqrt{\frac{2}{n-1}}\;\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

THEREFORE,

$$c = \frac{\theta}{\theta\sqrt{\frac{2}{n-1}}\;\frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)}}$$

$$c = \sqrt{\frac{n-1}{2}}\;\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

(b). $\bar{X}$ IS INDEPENDENT OF S^2

$$E\,\frac{\bar{X}}{S} = \left(E\,\bar{X}\right)\left(E\,S^{-1}\right)$$

WE KNOW $E\bar{X} = \mu$. AND $S^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$

$$E\,S^{-1} = E\left(S^2\right)^{-1/2} = \left(\frac{2\sigma^2}{n-1}\right)^{-1/2} \frac{\Gamma\left(\frac{n-1}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$E\,S^{-1} = \frac{1}{\sigma}\sqrt{\frac{n-1}{2}}\;\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

THEREFORE, $E\,\frac{\bar{X}}{S} = \frac{\mu}{\sigma}\sqrt{\frac{n-1}{2}}\;\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$

BUT WE WANT $E\,\frac{\bar{X}}{S} = \frac{\mu}{\sigma}$

SO THE UNBIASED ESTIMATOR OF $\frac{\mu}{\sigma}$ WILL BE:

$$\frac{\bar{X}}{S}\sqrt{\frac{2}{\sqrt{n-1}}}\;\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}.$$

(c). $M_{\bar{X}, X_1-\bar{X}}(t,s) = E\, e^{t\bar{X} + s(X_1-\bar{X})}$

$= E\, e^{X_1\left(\frac{t}{n}+s-\frac{s}{n}\right) + X_2\left(\frac{t}{n}-\frac{s}{n}\right)+\cdots+X_n\left(\frac{t}{n}-\frac{s}{n}\right)}$

BUT $X_1, \ldots, X_n$ ARE INDEPENDENT

$= M_{X_1}\left(\frac{t}{n}+s-\frac{s}{n}\right)\cdot\left\{M_{X_i}\left(\frac{t}{n}-\frac{s}{n}\right)\right\}^{n-1}$

$= e^{\mu\left(\frac{t}{n}+s-\frac{s}{n}\right)+\frac{1}{2}\left(\frac{t}{n}+s-\frac{s}{n}\right)^2\sigma^2} \cdot \left(e^{\mu\left(\frac{t}{n}-\frac{s}{n}\right)+\frac{1}{2}\left(\frac{t}{n}-\frac{s}{n}\right)^2\sigma^2}\right)^{n-1} = *$

NOW WE NEED TO SHOW THAT THIS IS EQUAL

TO $M_{\bar{X}}(t)\cdot M_{X_1-\bar{X}}(s)$

$\bar{X}\sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$, $\quad X_1-\bar{X} \sim N\left(0, \sqrt{\frac{n-1}{n}}\,\sigma\right)$

SO, $M_{\bar{X}}(t)\cdot M_{X_1-\bar{X}}(s) = e^{\mu t+\frac{1}{2}t^2\frac{\sigma^2}{n}} \cdot e^{0t+\frac{1}{2}s^2\frac{n-1}{n}\sigma^2}$

$* =$

(d). $E\left\{x'\left(I-\frac{1}{n}11'\right)x\right\}$

$$= E\left\{tr\, x'\left(I-\frac{1}{n}11'\right)x\right\}$$

$$= E\left\{tr\left(I-\frac{1}{n}11'\right)xx'\right\}$$

$$= tr\left\{\left(I-\frac{1}{n}11'\right)E\,xx'\right\}$$

<u>NOTE</u>: $E\,xx' = \Sigma + \mu\mu'$

$$= tr\left\{\left(I-\frac{1}{n}11'\right)\left(\sigma^2 I + \mu^2 11'\right)\right\}$$

$$= \sigma^2\, tr\left(I-\frac{1}{n}11'\right) + 0$$

$$= \sigma^2(n-1)$$

(e). $P(6 < \bar{x} < 10) \cdot P(0 < s^2 < 41)$

$\bar{x} \sim N(8,1)$

$$= (0.95)\, P\left(\frac{0.24}{25} < \frac{24 s^2}{25} < \frac{41(24)}{25}\right)$$

$$= (0.95)\, P\left(0 < \chi^2_{24} < 39.36\right)$$

$$= (0.95)(0.975) \approx 0.92625$$

PROBLEM 3 :

(a). $X_1 = Y_1 + Y_2$

$X_2 = Y_1 - Y_2$

$X_1 + X_2 = 2Y_1 \quad \rightarrow \quad Y_1 = \frac{X_1 + X_2}{2}$

AND $Y_2 = X_1 - Y_1 = X_1 - \frac{X_1 + X_2}{2}$

OR $Y_2 = \frac{X_1 - X_2}{2}$

THEREFORE, $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \underset{\sim}{X}$ BUT $X \sim N_2(\underset{\sim}{0}, I_2)$

SO $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N_2(A\mu, AA')$

WHERE $\underset{\sim}{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$A\mu = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underset{\sim}{0}$

$AA' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

$\underset{\sim}{Y} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\right)$

(b). $Y'\Sigma^{-1}Y = (y_1 \quad y_2)\begin{pmatrix}\sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}y_1 \\ y_2\end{pmatrix}$

$$= (y_1 \quad y_2)\begin{pmatrix}\dfrac{\sigma_2^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & -\dfrac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} \\ \dfrac{-\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & \dfrac{\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\end{pmatrix}\begin{pmatrix}y_1 \\ y_2\end{pmatrix}$$

$$= \left(\frac{y_1\sigma_2^2 - y_2\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}, \quad \frac{-y_1\sigma_{12} + y_2\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\right)\begin{pmatrix}y_1 \\ y_2\end{pmatrix}$$

$$= \frac{y_1^2\sigma_2^2 - y_1y_2\sigma_{12} - y_1y_2\sigma_{12} + y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$$

$$= \frac{y_1^2\sigma_2^2 - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2} = \frac{y_1^2\sigma_2^2 - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

Now: $Y'\Sigma^{-1}Y - \dfrac{y_1^2}{\sigma_1^2} = \dfrac{y_1^2\sigma_2^2 - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2 - \sigma_2^2(1-\rho^2)y_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$

$$= \frac{\cancel{y_1^2\sigma_2^2} - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2 - \cancel{y_1^2\sigma_2^2} + \rho^2y_1^2\sigma_2^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

$$= \frac{1}{1-\rho^2}\left[\frac{\rho^2y_1^2}{\sigma_1^2} - \frac{2y_1y_2\rho}{\sigma_1\sigma_2} + \frac{y_2^2}{\sigma_2^2}\right] = \frac{1}{1-\rho^2}\left(\frac{\rho y_1}{\sigma_1} - \frac{y_2}{\sigma_2}\right)^2$$

$$= \frac{Q^2}{1-\rho^2}, \text{ where } Q \sim N(0, \sqrt{1-\rho^2})$$

$$\therefore \sim \chi^2_1$$

(c). $\ln Y = b_0 + b_1 \cancel{X_i} + \varepsilon_i \quad \varepsilon_i \sim N(0, \sigma)$

$$M_{\ln Y}(t) = e^{(b_0 + b_1 X)t + \frac{1}{2}t^2\sigma^2} \qquad \ln Y \sim N(b_0 + b_1 X, \sigma)$$

Also

$$M_{\ln(Y)}(t) = E e^{t \ln Y} = E e^{\ln Y^t} = E Y^t$$

$$E Y^t = e^{(b_0 + b_1 X)t + \frac{1}{2}t^2\sigma^2}$$

$$\text{set } t = 1 \Rightarrow E Y = e^{b_0 + b_1 X + \frac{1}{2}\sigma^2}$$

$$\text{set } t = 2 \Rightarrow E Y^2 = e^{2(b_0 + b_1 X) + 2\sigma^2}$$

$$\operatorname{var}(Y) = E Y^2 - (E Y)^2$$

$$= e^{2(b_0 + b_1 X) + 2\sigma^2} - e^{2(b_0 + b_1 X) + \sigma^2}$$

$$= e^{2(b_0 + b_1 X) + \sigma^2}\left(e^{\sigma^2} - 1\right)$$

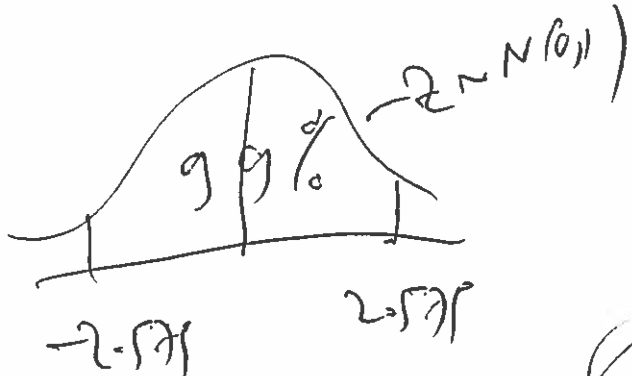
(d). $\bar{Y}_1 - \bar{Y}_2 \sim N\left(0, \sigma\sqrt{\frac{2}{n}}\right)$

$$P\left(|\bar{Y}_1 - \bar{Y}_2| < \frac{\sigma}{5}\right) = 0.99$$

$$P\left(-\frac{\sigma}{5} < \bar{Y}_1 - \bar{Y}_2 < \frac{\sigma}{5}\right) = 0.99$$

$$P\left(-\frac{\sigma/5}{\sigma\sqrt{\frac{2}{n}}} < \frac{\bar{Y}_1 - \bar{Y}_2 - 0}{\sigma\sqrt{\frac{2}{n}}} < +\frac{\sigma/5}{\sigma\sqrt{\frac{2}{n}}}\right) = 0.99$$

$$P\left(-\frac{\sqrt{n}}{\sqrt{10}} < Z < \frac{\sqrt{n}}{\sqrt{10}}\right) = 0.99$$



$$\frac{\sqrt{n}}{\sqrt{10}} = 2.575$$

$$n \approx 67$$

PROBLEM 4:

(a). $\bar{X}-\bar{Y} \sim N\left(\mu_1-\mu_2, \sqrt{\frac{\sigma_1^2}{9}+\frac{\sigma_2^2}{12}}\right)$

OR $\bar{X}-\bar{Y} \sim N\left(\mu_1-\mu_2, \sigma_2\sqrt{\frac{1}{3}+\frac{1}{12}}\right)$

$$\frac{8S_1^2}{\sigma_1^2}+\frac{11S_2^2}{\sigma_2^2} \sim \chi^2_{19}$$

OR $\dfrac{\frac{8}{3}S_1^2}{\sigma_2^2}+\dfrac{11S_2^2}{\sigma_2^2} \sim \chi^2_{19}$

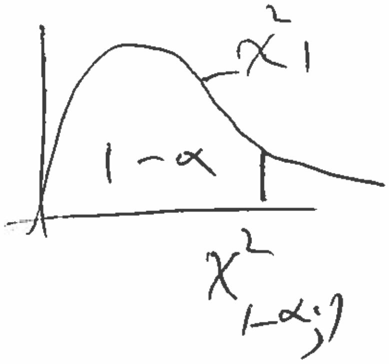
FORM A t STATISTIC:

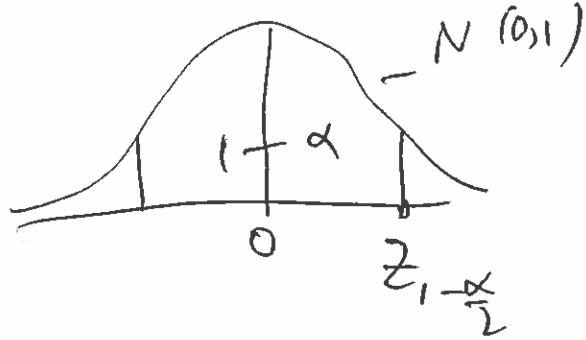
$$\frac{\dfrac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\cancel{\sigma_2}\sqrt{\frac{1}{3}+\frac{1}{12}}}}{\sqrt{\dfrac{\frac{8}{3}S_1^2+11S_2^2}{\cancel{\sigma_2^2}}\Big/19}} \sim t_{19}$$

$$\frac{\sqrt{19}\sqrt{3}\sqrt{12}}{\sqrt{5}} \quad \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{8S_1^2+33S_2^2}} \sim t_{19}$$

$$\frac{\sqrt{684}}{\sqrt{5}} \quad \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{8S_1^2+33S_2^2}} \sim t_{19}$$

(b). $X = Z^2$ $\quad Z \sim N(0,1)$, $\quad X \sim \chi^2_1$





(or $Z_{\frac{\alpha}{2}}$)

$$Z^2_{\frac{\alpha}{2}} = \chi^2_{1-\frac{\alpha}{2};1}$$

<u>EXAMPLE</u>: $1-\alpha = 0.90$ $\quad \chi^2_{0.90;1} = 2.71$

$Z_{0.05} = 1.645$ AND $1.645^2 = 2.71$

(c). $X \sim b(n,p)$ $\qquad \hat{p} = \frac{X}{n}$ $\qquad E\hat{p} = p$

$$E\, n\hat{p}(1-\hat{p}) = nE\hat{p} - nE\hat{p}^2$$

$$= np - n\left(\mathrm{VAR}(\hat{p}) + (E\hat{p})^2\right)$$

$$= np - n\left(\mathrm{VAR}\left(\frac{X}{n}\right) + p^2\right)$$

$$= np - n\left(\frac{np(1-p)}{n^2} + p^2\right)$$

$$= np - p(1-p) - np^2$$

$$= p(1-p)(n-1) =$$

WE WANT $E c\hat{\theta} = np(1-p)$

$$c(n-1)p(1-p) = np(1-p)$$

$$C = \frac{n}{n-1},$$

THEREFORE UNBIASED WILL BE:

$$\hat{\theta}_1 = \frac{n}{n-1}\, n\hat{p}(1-\hat{p}).$$

(d). $X_1 \sim N(1,1)$, $X_2 \sim N(2,2)$, $X_3 \sim N(3,3)$

(i). $\left(\frac{X_1-1}{1}\right)^2 + \left(\frac{X_2-2}{2}\right)^2 + \left(\frac{X_3-3}{3}\right)^2 \sim \chi^2_3$

(ii). $$\frac{\frac{X_1-1}{1}}{\sqrt{\left(\frac{X_2-2}{2}\right)^2+\left(\frac{X_3-3}{3}\right)^2 / 2}} \sim t_2$$

(iii). $$\frac{\left(\frac{X_1-1}{1}\right)^2 / 1}{\left\{\left(\frac{X_2-2}{2}\right)^2+\left(\frac{X_3-3}{3}\right)^2\right\} / 2} \sim F_{1,2}$$

(e). $X_1 - X_2 \sim N(0, \sigma\sqrt{2})$

$$\left(\frac{X_1-X_2-0}{\sigma\sqrt{2}}\right)^2 \sim \chi^2_1$$

$$E\left(\frac{X_1-X_2-0}{\sigma\sqrt{2}}\right)^2 = 1 \Rightarrow E\,\frac{(X_1-X_2)^2}{2} = \sigma^2$$

YES.

$$\mathrm{VAR}\left(\frac{X_1-X_2-0}{\sigma\sqrt{2}}\right)^2 = 2$$

$$\mathrm{VAR}\,(X_1-X_2)^2 = 4\sigma^2$$
