STATS 100A Midterm Solution

Problem 1 Suppose we flip a fair coin 100 times independently. Let X be the number of heads. Calculate $\mathrm{E}(X)$, $\mathrm{Var}(X)$, $SD(X)$, $\mathrm{E}(X/100)$, $\mathrm{Var}(X/100)$, $SD(X/100)$. Write down the formula for computing $P(X \in [40, 60])$.

A:

$$\begin{aligned}
&\mathrm{E}(X) = np = 100 \times (1/2) = 50\\
&\mathrm{Var}(X) = np(1-p) = 100 \times (1/2) \times (1 - 1/2) = 25\\
&SD(X) = \sqrt{\mathrm{Var}(X)} = 5\\
&\mathrm{E}(X/100) = \mathrm{E}(X)/100 = 1/2\\
&\mathrm{Var}(X/100) = \mathrm{Var}(X)/100^2 = 2.5 \times 10^{-3}\\
&SD(X/100) = SD(X)/100 = .05\\
&P(X \in [40, 60]) = \sum_{k=40}^{60} P(X = k) = \sum_{k=40}^{60} \binom{100}{k} / 2^{100}
\end{aligned}$$

Problem 2 Suppose within the population of voters, 20% of them support a candidate A. If we randomly sample 100 people sequentially with replacement. Let X be the number of supporters of A among these 100 people. Then what is the distribution of X? What are $\mathrm{E}(X)$, $\mathrm{Var}(X)$, and $SD(X)$? What are $\mathrm{E}(X/100)$, $\mathrm{Var}(X/100)$, and $SD(X/100)$? A:

$$\begin{aligned}
&X \sim \text{Binomial}(n = 100, p = .2)\\
&\mathrm{E}(X) = np = 100 \times .2 = 20\\
&\mathrm{Var}(X) = np(1-p) = 100 \times .2 \times (1 - .2) = 16\\
&SD(X) = \sqrt{\mathrm{Var}(X)} = 4\\
&\mathrm{E}(X/100) = \mathrm{E}(X)/100 = .2\\
&\mathrm{Var}(X/100) = \mathrm{Var}(X)/100^2 = 1.6 \times 10^{-3}\\
&SD(X/100) = SD(X)/100 = .04
\end{aligned}$$

Problem 3 Suppose we randomly throw 10,000 points into the unit square $[0, 1]^2$. Let A be the region $x^2 + y^2 \leq 1$. Let m be the number of points that fall into A. What is the distribution of m? Let $\hat{\pi} = 4m/10000$ be our Monte Carlo estimate of π. What are $\mathrm{E}(\hat{\pi})$, $\mathrm{Var}(\hat{\pi})$ and $SD(\hat{\pi})$? A:

$$\begin{aligned}
&m \sim \text{Binomial}(n = 10^4, p = \pi/4)\\
&\mathrm{E}(\hat{\pi}) = \mathrm{E}(4m/10^4) = (4/10^4)\mathrm{E}(m) = (4/10^4) \times 10^4 \times \pi/4 = \pi\\
&\mathrm{Var}(\hat{\pi}) = \mathrm{Var}(4m/10^4) = (4/10^4)^2\mathrm{Var}(m) = (4/10^4)^2 \times 10^4 \times \pi/4 \times (1 - \pi/4) = \pi(4-\pi)/10^4.\\
&\mathrm{Var}(\hat{\pi}) = \sqrt{\pi(4-\pi)}/10^2.
\end{aligned}$$