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Stats 100 A Final

Problem 1

$$f(x) = \begin{cases} ax^2 & , x \in [0,1] \\ 0 & , x \notin [0,1] \end{cases}$$

1) $a = ?$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$ // property of PDF

$\int_0^1 ax^2\,dx = 1$

$a\int_0^1 x^2\,dx = 1$

$a\left[\frac{x^3}{3}\right]_0^1 = 1$

$\frac{a}{3} = 1$

$a = 3$

2) $P(x \geq 1/2) = \int_{1/2}^1 f(x)\,dx$

$\int_{1/2}^1 3x^2\,dx$

$\left[x^3\right]_{1/2}^1$

$(1)^3 - (\frac{1}{2})^3 = \frac{7}{8}$

$P(x \geq 1/2) = \frac{7}{8}$

3) $E(x)$, $Var(x)$

$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 3x^3\,dx$

$\frac{3}{4}\left[x^4\right]_0^1 = \frac{3}{4}$

$E(x) = \frac{3}{4}$

$Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 3x^4\,dx$

$\frac{3}{5}\left[x^5\right]_0^1$

$E(x^2) = \frac{3}{5}$

$Var(x) = \frac{3}{5} - (\frac{3}{4})^2 = \frac{3}{80}$

$E(x) = \frac{3}{4}$ $\quad Var(x) = \frac{3}{80}$

Problem 1 continued

4) $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $x_i \sim F(x)$, $n = 100$

let $\mu = E(x_i)$ and $\sigma^2 = Var(x_i)$

$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}(n\mu) = \mu$

$E(\bar{x}) = \mu = \frac{3}{4}$

$Var(\bar{x}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(x_i) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$

if $n=100$ $Var(\bar{x}) = \frac{\sigma^2}{n} = \frac{3/80}{100} = \frac{3}{8000}$

$E(\bar{x}) = \mu = \frac{3}{4}$ stated in part 5

$Var(\bar{x}) = \frac{\sigma^2}{n} = \frac{3}{8000}$ // if $n=100$

By the Law of Large numbers, $\bar{x} \to \mu$

$\therefore$ $\bar{x}$ will converge to $\frac{3}{4}$

5) 95% = $[\mu - 2\sigma, \mu + 2\sigma]$

$\bar{x}$ must follow : $\bar{x} \sim Normal\left(\mu, \frac{\sigma^2}{n}\right)$

$\bar{x} \sim Normal\left(\frac{3}{4}, \frac{3}{8000}\right)$

$\sigma = \sqrt{Var(\bar{x})} = \sqrt{\frac{3}{8000}} = .0194$

$\mu = \frac{3}{4} = .75$

$[\mu - 2\sigma, \mu + 2\sigma] = [.7112, .7887]$

Problem 2

$\Delta t = \frac{t}{n}$, $x_0 = 0$

$x_t = \sum_{i=1}^{n} z_i$

$z_i = \mu \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon_i$

ε_i are independent , $E(\varepsilon_i) = 0$, $Var(\varepsilon_i) = 1$

1) $E(z_i)$, $Var(z_i)$

$E(z_i) = E(\mu \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon_i)$

$E(z_i) = \mu \Delta t + \cancel{\sigma \sqrt{\Delta t}\, E(\varepsilon_i)}^{\,0}$ // $E(\varepsilon_i) = 0$

$E(z_i) = \mu \Delta t$

$Var(z_i) = Var(\mu \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon_i)$

$Var(z_i) = \sigma^2 \Delta t \, Var(\varepsilon_i)$ // $Var(\varepsilon_i) = 1$

$Var(z_i) = \sigma^2 \Delta t$

$E(z_i) = \mu \Delta t$ $\quad Var(z_i) = \sigma^2 \Delta t$

2) $E(X_t)$, $Var(X_t)$

$E(X_t) = E(\sum_{i=1}^{n} z_i) = \sum_{i=1}^{n} E(z_i) = n(\mu \Delta t)$

$E(X_t) = n(\mu \Delta t)$ // $\Delta t = \frac{t}{n}$

$E(X_t) = n\mu(\frac{t}{n})$

$E(X_t) = \mu t$

$Var(X_t) = Var(\sum_{i=1}^{n} z_i) = \sum_{i=1}^{n} Var(z_i) = n(\sigma^2 \Delta t)$

$Var(X_t) = n\sigma^2 \Delta t = n\sigma^2(\frac{t}{n})$ // $\Delta t = \frac{t}{n}$

$Var(X_t) = \sigma^2 t$

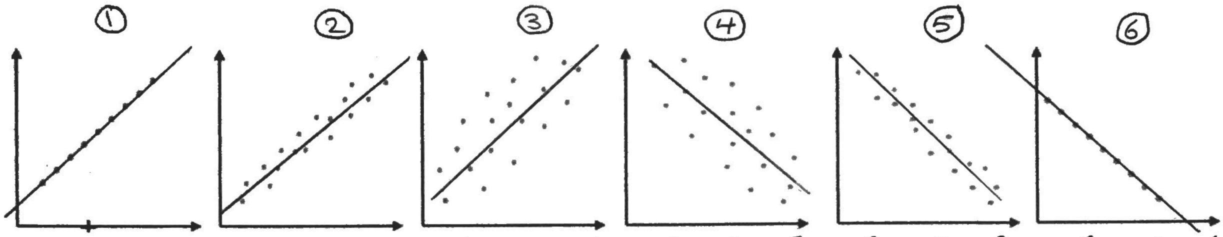
// Both $E(X_t)$ and $Var(X_t)$ do NOT depend on n

$E(X_t) = \mu t$ $\quad Var(X_t) = \sigma^2 t$.

3) For large n, X_t will follow a normal distribution

$X_t \sim Normal(\mu t, \sigma^2 t)$

Problem 3



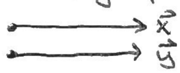
Corr: +1 Corr : +.9 Corr : +.5 Corr : -.5 Corr : -.9 Corr : -1

1) correlation labeled on graph

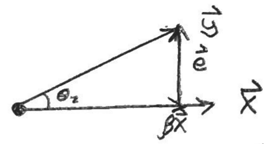
2) let θ_i be the angle between $\vec{x}, \vec{y}$
let $\beta = \text{Corr}(X, Y)$

plot 1: $\theta_1 = 0$

$\beta = 1 \therefore \vec{x} = \vec{y} = \beta\vec{x}$
direction($\vec{x}$) = direction($\vec{y}$)
$\theta = 0$
visual : $\vec{x}$ and $\vec{y}$ having different starting positions

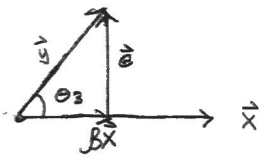


plot 2 : $\theta_2 \approx 25.8°$



$\beta = .9$

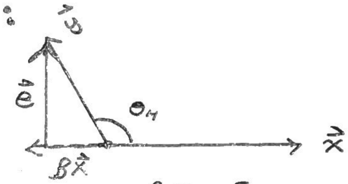
plot 3:
$\theta_3 = 60°$



$\beta = .5$

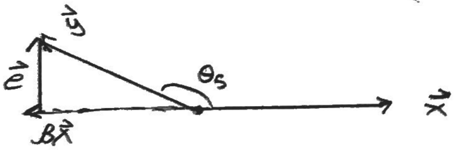
plot 4:

$\theta_4 = 120°$



$\beta = -.5$

plot 5: $\theta_5 \approx 154.2°$



$\beta = -.9$

plot 6 : $\theta_6 = 180°$

$\beta = -1 \therefore \vec{y} = -\vec{x}$
and $|\vec{e}| = 0$

3) • in plot 1 and plot 6 $|\vec{e}| = 0$ due to perfect correlation also $\sin(\theta_1) = \sin(\theta_6) = 0$

- in plot 1 $\vec{x} = \vec{y} = \beta\vec{x}$ because $\beta = 1$ and since $\theta_1 = 0°$ $\vec{x}$ and $\vec{y}$ can be drawn as the same vector

$\tilde{y} = \beta\tilde{x}$ shown on each Scatterplot