~~12.5/15~~ 13.5/16

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MUST DO BEFORE STARTING EXAM

UCLA ID: [redacted] **TA Session:** [redacted]

LAST NAME (Please, PRINT): [redacted] **– (Put ID on your desk)**

FIRST NAME: [redacted]

ENGLISH SIGNATURE: [redacted]

(a) NO GRAPHICS CALCULATORS ALLOWED.

(b) WRITE YOUR NAME ON ALL SIDES OF THE CHEAT SHEET, TOP RIGHT HAND CORNER. KEEP NORMAL TABLES AND TABLE OF DISTRIBUTIONS STAPLED TO THE CHEAT SHEET. THE CHEAT SHEET CAN HAVE ONLY FORMULAS AND DEFINITIONS. NO SOLVED PROBLEMS, NO NUMERICAL EXAMPLES, NO INTERMEDIATE STEPS, NO PROOFS, NO EXAMPLES.

(c) ALL YOUR BELONGINGSMUST BE UNDER THE CHAIR.

(d) PUT DOWN THE TABLES ON YOUR RIGHT AND LEFT. ALL ITEMS MUST BE ON YOUR DESK.

(e) SHARING INFORMATION, TALKING IN THE EXAM ROOM, LOOKING AT EACH OTHER'S EXAMS AND USING YOUR ELECTRONIC DEVICES IS NOT ALLOWED.

(f) ALL ELECTRONIC DEVICES MUST BE INSIDE YOUR BACKPACK AND NOT ACCESSED UNTIL YOU LEAVE THE ROOM.

(g) DO NOT TALK IN THE EXAM ROOM AT ANY TIME. WAIT UNTIL YOU ARE OUTSIDE.

(h) THE EXAM MUST REMAIN STAPLED THROUGHOUT THE WHOLE EXAM.

ONLY ANSWERS WRITTEN IN THE SPACE PROVIDED AFTER EACH QUESTION WILL BE READ. WORK WRITTEN ELSWEHERE WILL NOT BE READ.

Problem 1. The cumulative distribution function of a random variable is given by

$$F(x) = 1 - (1 - p)^x, \qquad x = 1, 2, 3, \ldots\ldots\ldots$$

Does this random variable have the memoryless property? Whatever your answer is, prove it. Show work.

$$P(X > s+t) = 1 - F(s+t)$$
$$= 1 - [1 - (1-p)^{s+t}]$$
$$= (1-p)^{s+t}$$
$$P(X > s) = 1 - F(s)$$
$$= 1 - (1 - (1-p)^s] = (1-p)^s$$
$$P(X > t) = 1 - F(t)$$
$$= 1 - [1 - (1-p)^t] = (1-p)^t$$

Hence, $P(X > s)P(X > t) = (1-p)^s(1-p)^t$
$$= (1-p)^{s+t}$$
$$= P(X > s+t).$$

Thus, R.V. has memoryless property

Problem 2. (1 pt) The service times $X_1, X_2, \ldots.., X_5$, of 5 teller windows in a bank are independent and identically distributed random variables. Each X_i is exponential with an expected value of 5 minutes. In a given day, 5 independent customers arrive at 9:00 am and each is serviced by a different window.

What is the probability that all 5 customers will be there at 9:07? Show work.

$X_i \sim$ Exponential $(\lambda = 1/5)$

$$P(X_i > 7) = 1 - P(X_i < 7)$$
$$= 1 - [1 - e^{-7/5}]$$
$$= e^{-7/5}$$

The probability that all 5 will be there

$$= \prod_{i=1}^{5} P(X_i > 7) = (e^{-7/5})^5 = \underline{\underline{e^{-7}}}$$

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Problem 3. From a box containing four white and three red balls, two balls are selected at random, without replacement. Find the probabilities of the following events.

(a) Exactly one white ball is selected. Show work.

$$P(1 \text{ white ball selected}) = \frac{\binom{4}{1}\binom{3}{1}}{\binom{7}{2}} = \frac{(4)(3)2}{7(6)} = \left(\frac{4}{7}\right) = 0.571$$

(b) At least one white ball is selected. Show work.

$$P(\text{at least } 1 \text{ white ball selected}) = 1 - P(\text{no white ball selected}) = 1 - \frac{\binom{3}{2}}{\binom{7}{2}} = 1 - \left(\frac{3}{21}\right) = \frac{6}{7} = 0.857$$

(c) The second ball drawn is white. Show work.

$$P(\text{second ball white}) = P(\text{second ball white} \cap \text{1st ball white}) + P(\text{second ball white} \cap \text{1st ball red})$$

$$= \left(\frac{(3)(4)}{2!\binom{7}{2}}\right) + \left(\frac{(3)(4)}{2!\binom{7}{2}}\right) = \frac{12}{42} + \frac{12}{42} = \frac{24}{42} = \frac{4}{7} = 0.571$$

Problem 4. The Standard Normal distribution is just a normal distribution with $\mu = 0$ and $\sigma^2 = 1$. It turns out that if we standardize a Normal random variable X with parameters μ, σ in the following way

$$z = \frac{X - \mu}{\sigma}$$

The expected value of the transformation is 0 and the variance is 1. Prove it. Show work.

$$E(z) = E\left(\frac{X-\mu}{\sigma}\right) = E\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right) = \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$

$$Var(z) = Var\left(\frac{X-\mu}{\sigma}\right) = Var\left(\frac{X}{\sigma}\right) - Var\left(\frac{\mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X) - 0 = \frac{\sigma^2}{\sigma^2} = 1$$

Stat 100A -Intro Probability
Midterm 2 Spring 2017

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Problem 5. The weekly downtime Y(in hours) for a certain industrial machine has approximately a gamma distribution with $\alpha = 3$ and $\lambda = 2$. The loss L (in dollars) to the industrial operation as a result of the downtime is given by

$$L = 30Y + 2Y^2$$

What is the expected weekly loss due to downtime? Show work.

$E(L) = E(30Y + 2Y^2)$

$= 30E(Y) + 2E(Y^2)$

Now for $Y \sim \text{Gamma}(\alpha = 3, \lambda = 2)$, $E(Y) = \frac{\alpha}{\lambda} = 3/2$

Also, m.g.f $= M_Y(t) = (1 - \lambda^{-1}t)^{-\alpha}$

Hence, $M_Y'(t) = \left(\frac{+\alpha}{\lambda}\right)(1-\lambda^{-1}t)^{-\alpha-1}$ and $M_Y''(t) = \frac{\alpha(\alpha-1)}{\lambda^2}(1-\lambda^{-1}t)^{-\alpha-2}$

Thus, $E(Y^2) = M_Y''(0) = \frac{\alpha(\alpha-1)}{\lambda^2} = \frac{6}{4} = 3/2$. $E(L) = 32(3/2) = \boxed{48}$

Problem 6. The proportion of air pollution particles in the air of a urban area has the following probability density function

$$f(y) = 1, \qquad 0 \le y \le 1$$

(a) Let Y_1, Y_2, Y_3, Y_4 be the proportion in four independent urban areas, each of them with the density function given above. Find the moment generating function and the expected value and variance of $W = Y_1 + Y_2 + Y_3 + Y_4$. Show work.

$W = Y_1 + Y_2 + Y_3 + Y_4$.

Now, $M_{Y_i}(t) = E(e^{tx}) = \int_0^1 e^{tx} f(x)\,dx = \int_0^1 e^{tx}\,dx$

$= \frac{1}{t} e^{tx}\Big|_0^1 = \frac{1}{t}(e^t -$

what's $M_W(t)$?

$E(W) = \sum_{i=1}^{4} E(Y_i) = 4\left[\int_0^1 x\,dx\right] = 4\left(\frac{x^2}{2}\right)\Big|_0^1 = 4\left(\frac{1}{2}\right) = 2$

-1.5

$Var(W) = \sum_{i=1}^{4} Var(Y_i) = 4\int_0^1 \left(x - \frac{1}{2}\right)^2 dx = 4\left(x^2 + \frac{1}{4} - x\right)\Big|_0^1$

$= 4\left(1 + \frac{1}{4} - 1\right) = 1$

(b) What is the probability that in a sample of 50 cities, more than 20 of them have a proportion of air pollution particles smaller than 0.2? Show work.

$P(Y_i < 0.2) = \int_0^{0.2} dy = (0.2)$

Let T = # of cities with particles smaller than 0.2.

Now, $T \sim \text{Binomial}(n = 50, p = 0.2)$

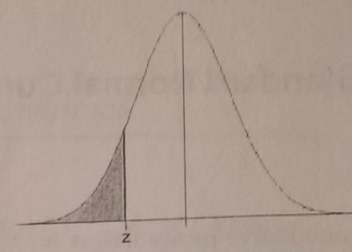
Thus, $np \ge 10$ and $n(1-p) \ge 10$. Hence, we can use normal approximation with $\mu = np = 10$ and $\sigma^2 = np(1-p) = 10(0.8) = 8$

Thus, $P(X > 20) = 1 - P(X \le 20) = 1 - P\left(Z \le \frac{20 - 10}{\sqrt{8}}\right)$

$= 1 - P\left(Z \le \frac{10}{\sqrt{8}}\right) = 1 - P(Z \le 3.5$

-2.5

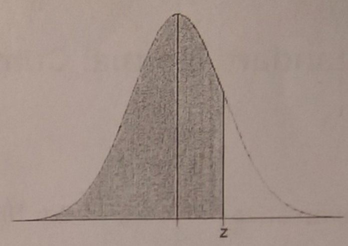
Standard Normal Cumulative Probability Table



Cumulative probabilities for NEGATIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

TABLE 7.1: DISCRETE PROBABILITY DISTRIBUTION

	Probability mass function, $p(x)$	Moment generating function, $M(t)$	Mean	Variance
Binomial with parameters n, p; $0 \leq p \leq 1$	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, \ldots, n$	$(pe^t + 1 - p)^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$ $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \leq p \leq 1$	$p(1-p)^{x-1}$ $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative binomial with parameters r, p; $0 \leq p \leq 1$	$\binom{n-1}{r-1} p^r (1-p)^{n-r}$ $n = r, r+1, \ldots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

TABLE 7.2: CONTINUOUS PROBABILITY DISTRIBUTION

	Probability mass function, $f(x)$	Moment generating function, $M(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters $(s, \lambda), \lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^s$	$\frac{s}{\lambda}$	$\frac{s}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2
Chisquare with n degrees of freedom	a gamma with parameters $\lambda = 1/2$ $\alpha = \frac{n}{2}$	$(1-2t)^{n/2}$	n	$2n$