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Exam 1
21 October 2011

$\frac{90}{100}$

Problem 1 (20 points)
Answer the following questions:

a. Let X follow the Poisson distribution with parameter λ. Show that

$$EX^n = \lambda E(X+1)^{n-1}. \quad \textit{Hint: } \text{Write } X^n = XX^{n-1} \text{ and then find } EXX^{n-1}.$$

Then use this result to find EX^3.

$X^n = XX^{n-1}$

$EXX^{n-1} = \sum_x xx^{n-1} P(x)$

$= \sum_{x=0}^{\infty} xx^{n-1} \frac{\lambda^x e^{-\lambda}}{x!}$

$= \sum_{x=0}^{\infty} x^{n-1} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$

$= \lambda \sum_{x=0}^{\infty} x^{n-1} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!}$ $y = x-1$, $x = y+1$

$= \lambda \sum_{y=-1}^{\infty} (y+1)^{n-1} \frac{\lambda^y e^{-\lambda}}{y!}$

~~therefore~~ $= \lambda E(x+1)^{n-1}$

$\boxed{\text{therefore } Ex^n = \lambda E(x+1)^{n-1}}$

$\boxed{Ex^3 = \lambda E(x+1)^2} = \lambda E(x^2+2x+1)$

$Ex^3 = \lambda E(x+1)^2$

$Ex = \lambda$

$E(x+1)^{n-1} = \sum_x (x+1)^{n-1} P(x)$

$= \sum_{x=0}^{\infty} (x+1)^{n-1} \frac{\lambda^x e^{-\lambda}}{x!}$

$= \sum_{x=0}^{\infty} (x+1)^{n-1} \frac{\lambda^x e^{-\lambda}}{x!}$

$= \underbrace{\sum_{x=-1}^{\infty} (x+1)^{n-1} \frac{\lambda^x e^{-\lambda}}{x!}}$

when $x=-1$
$(x+1)=0$

$x+1=0$
$-1+1=0$

$\lambda(Ex^2+2Ex+1)$
$= \lambda(\lambda^2+\lambda+2\lambda+1)$

from class
$Ex^2 = \lambda^2+\lambda$

$\boxed{Ex^3 = \lambda^3+3\lambda^2+\lambda}$

b. Suppose there are n people in a room. Compute or explain how to find the minimum value of n so that the probability that no 4 of them have the same birthday is less than $\frac{1}{2}$ using the Poisson distribution.

$p = \frac{\binom{365}{1}}{\binom{365}{1}^4} = \frac{1}{365^3}$ ~~$P(X=0)$~~

~~no four = 1 - 1/365³~~

$\binom{n}{4}$ groups of four

$\lambda = \binom{n}{4} \frac{1}{365^3}$

$P(X=0) = \frac{\lambda^0 e^{-\lambda}}{x!} = e^{-\lambda} = e^{-\binom{n}{4}\left(\frac{1}{365^3}\right)} \le .5$

$-\binom{n}{4}\left(\frac{1}{365^3}\right) \le \ln \frac{1}{2}$

$\binom{n}{4}\left(\frac{1}{365^3}\right) \ge \ln 2$

$\frac{n!}{4!(n-4)!} \ge 365^3 \ln 2$

$\frac{n!}{(n-4)!} \ge 4! \, 365^3 \ln 2$

$(n)(n-1)(n-2)(n-3) \ge 4! \, 365^3 \ln 2$

$n^4 - 6n^3 + 11n^2 - 6n \ge 8.089 \times 10^8$

$n \ge 170.3$

$\boxed{n = 171}$

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

Problem 2 (20 points)
Answer the following questions:

a. Compute the expected value and standard deviation of the *minimum* number when two dice are rolled. You will have to find the distribution of the minimum (X) first (i.e. complete the table below).

X	$P(X)$
1	11/36
2	9/36
3	7/36
4	5/36
5	3/36
6	1/36

$X=1 \quad P(X=1) = P(D_1=1 \cap D_2 \geq 1)$

$= P(D_1=1)P(D_2 \geq 1)$

$X=2 \quad P(X=2) = P(D_1=2 \cap D_2 \geq 2)$

$X=3 \quad P(X=3) = P(D_1=3 \cap D_2 \geq 3)$

$$EX = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + \frac{6}{36}$$

$$= \frac{91}{36} = \boxed{2.53}$$

$$\text{var}(X) = 1 \cdot \frac{11}{36} + 2^2 \frac{9}{36} + 3^2 \frac{7}{36} + 4^2 \frac{5}{36} + 5^2 \frac{3}{36} + \frac{6^2}{36} - 2.53^2$$

$$= 1.96$$

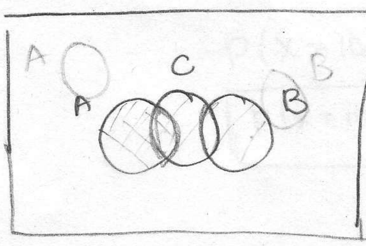
$$\text{Std dev.} = \sqrt{1.96} = \boxed{1.40}$$

$X=4 \quad P(X=4) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12} \times 2$

$X=5 \quad P(X=5) = \frac{1}{6} \cdot \frac{2}{6} = \frac{1}{18} \times 2$

$X=6 \quad P(X=6) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \times 2$

b. Suppose that A, B, and C are three events such that
A, B are disjoint,
A, C are independent, and
B, C are independent.
Suppose also that $4P(A) = 2P(B) = P(C)$, and $P(A \cup B \cup C) = 5P(A)$.
Determine the value of $P(A)$.



$= 5P(A)$

$P(A \cup B \cup C) = 5P(A)$

$4P(A) = 2P(B) = P(C)$

A, B disjoint so

$P(A \cup B) = P(A) + P(B)$

$P(B) = 2P(A)$

$P(A \cup B) = 3P(A)$

$P(A) = 1 - P(B \cup C) - P(A \cap C)$

$= 1 - (P(B) + P(C) - P(B \cap C)) - P(A)P(C)$

$= 1 - P(B) - P(C) + P(B)P(C) - P(A)P(C)$

$P(A \cup B \cup C) = 5P(A) \quad P(A \cup B) = 3P(A)$

$P(C) - P(A \cap C) - P(B \cap C) = 2P(A)$

$4P(A) - P(A \cap C) - P(B \cap C) = 2P(A)$

$P(A \cap C) + P(B \cap C) = 2P(A)$

$P(A)P(C) + P(B)P(C) = 2P(A)$

$P(A) + P(B) = \frac{2P(A)}{4P(A)} \quad P(B) = 2P(A)$

$\leftarrow$ assume $P(A) \neq 0$

$P(A) + 2P(A) = \frac{1}{2}$

$3P(A) = \frac{1}{2} \quad \boxed{P(A) = \frac{1}{6}}$

Problem 3 (20 points)
Answer the following questions:

a. Consider the casino roulette game. If a player bets $1 on a single game and we let X be the casino's profit we get the following probability distribution:

X	$P(x)$
-35	$\frac{1}{38}$
1	$\frac{37}{38}$

In 500 such games, what is the *exact* probability that the casino will make more than $200? You don't need to compute this probability, but you must write the exact expression.

Profit > 200
casino must win n games

$\cancel{-\frac{1}{38}\left(\frac{500}{38}\right) + \left(\frac{37 \cdot 500}{38}\right)}$

$n(1) + (500-n)(-35) > 200$

$n - 17500 + 35n > 200$

$36n > 17700$

$n > 491.6$

$n \geq 492$

out of 500, win at least 492

$\cancel{P(X \geq 492) = \binom{500}{492}\left(\frac{37}{38}\right)^{492}\left(\frac{1}{38}\right)^{8}}$

$$\boxed{P(X \geq 492) = \sum_{x=492}^{500} \binom{500}{x}\left(\frac{37}{38}\right)^{x}\left(\frac{1}{38}\right)^{500-x}}$$

b. The probability of receiving a one pair poker hand (for example A,A,4,5,10) is 42%. Find the probability that the 6th one pair poker hand for a player will be observed on his 10th game.

neg. bin

$P(X=10) = \binom{10-1}{6-1}(.42)^6(.57)^4$

$$\boxed{P(X=10) = \binom{9}{5}(.42)^6(.57)^4}$$

c. Let X follow the binomial distribution with parameters n and $p > 0$. Is it true that the mean is always larger than the standard deviation? If not, please find the condition under which this is not true.

$X \sim b(n, p)$ $\quad p > 0 \qquad \mu = np \quad \sqrt{\sigma} = \sqrt{np(1-p)}$

$np > \sqrt{np(1-p)}$

$n^2p^2 > np(1-p)$

$np > 1-p \qquad n > \frac{1-p}{p}$

$np + p > 1$

$p(n+1) > 1$

$p > \frac{1}{n+1}$

it is not always true that $p > \frac{1}{n+1}$, so the mean is not always larger than the std. dev. It is smaller, or equal when $p \leq \frac{1}{n+1}$

Problem 4 (20 points)
Answer the following questions:

a. A company rents luxurious cars to customers for recreational purposes. Customers must rent these cars for the entire day. A decision must be made about how many of these cars the company must have. They don't want to have too many cars but at the same time they don't want to turn customers away due to lack of cars availability. Suppose that from experience we know that the number of customers follows the Poisson distribution with $\lambda = 6.1$ per day. The owner of the company would like to find the minimum number of cars so that, with probability 90% or more, all the customers will get a car. Explain how you will be able to find the minimum number of cars. You don't need to submit a final answer as this will take long, but please be very specific. One should be able to find the number of cars by reading your solution!

$\lambda = 6.1$ per day $\qquad P(X=x) \geq .9$

$$P(X=x) = \frac{6.1^x e^{-6.1}}{x!} \geq .9$$

Find the values of x that satisfy this equation.
The smallest integer of x such that $\frac{6.1^x e^{-6.1}}{x!} \geq .9$ is the number of cars the company should keep.

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b. Refer to part (a). Suppose that the number of customers follows the Poisson distribution with $\lambda = 6.1$ per day. The probability that at any day at least nine customers will show up is 0.16. What is the probability that it will take more than 13 days until the company observes a day with at least nine customers? geo.

$p = .16 \qquad P(X > 13) = (1-p)^{13}$
$= (1-.16)^{13}$
$= .84^{13}$

$\boxed{P(X>13) = .104}$

$x \leq 16 - x \geq 6 \qquad$ 6 — 16 $\qquad x \leq 5$

d. Refer to part (b). What is the probability that it will take between 6 and 16 days (including) until the company observes a day with at least nine customers?

$P(6 \leq X \leq 16) = P(X \leq 16) - P(X \leq 6)$ $\qquad P(X \leq K) = 1-(1-p)^{K-1}$
$= 1-(1-p)^{16} - (1-(1-p)^5)$
$= -.84^{16} + (.84)^5$

$\boxed{P(6 \leq X \leq 16) = .357}$

Problem 5 (20 points)
Answer the following questions:

a. In a certain city, taxis are easily identified by rooftop lights. There are two taxi companies in this city, Green and Blue, with taxis colored accordingly. One night, there is a hit-and-run accident involving a taxi. The vehicle was clearly a taxi, as multiple witnesses saw the rooftop light. However, only one witness (Mr. A) claimed to be able to identify the taxi by color. Mr. A claimed that this was a Green taxi. Here are some important facts:
Mr. A was given an identification test under lighting conditions similar to those the night of the accident. When the taxi was blue, he identified it correctly 80% of the time. When the test taxi was green, he identified it correctly 80% of the time. A number of other people were given the same identification test, and they also produced the same 80% probability that Mr. A got. Also, we know that in this city, 85% of the taxis are Blue.

What is the probability that the accident was really committed by a Green taxi.

$P(G|IdG) = \frac{P(G \cap IdG)}{P(IdG)}$

$P(IdG) = P(IdG \cap B) + P(IdG \cap G)$

$= P(IdG|B)P(B) + P(IdG|G)P(G)$

$= .2(.85) + .8(.15) = .29$

$P(G|IdG) = \frac{.8(.15)}{.29} = .414$

probability really committed by Green is .414

b. A purchaser of electrical components buys them in lots of size 10. It is his policy to inspect 3 components randomly from a lot and to accept the lot only if all 3 components are nondefective. If 30 percent of the lots have 4 defective components and 70 percent have only 1, what proportion of lots does the purchaser reject? *Hint:* It will be easier first to compute the proportion of lots that the purchaser accepts.

$\cancel{P(A) = P(C_1 \cap C_2 \cap C_3)}$

$P(A) = P(A \cap 4def) + P(A \cap 1def)$

$= P(A|4def)P(4def) + P(A|1def)P(1def)$

$= .967(.3) + 1(.7)$

$= .9901$

$P(R) = 1 - P(A) = 1 - .9901 = \boxed{.0099}$

$P(A|4def) = \frac{\binom{6}{3} + \binom{4}{2}\binom{6}{1} + \binom{4}{1}\binom{6}{2}}{\binom{10}{3}}$

$= .967$

$P(A|1def) = \frac{\binom{1}{1}\binom{9}{2} + \binom{1}{0}\binom{9}{3}}{\binom{10}{3}}$

$= 1$

$P(R|4def) = \frac{\binom{4}{3}\binom{6}{0}}{\binom{10}{3}}$

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