Midterm 2

Friday, February 26, 2021 5:52 PM

You must show your work to receive credit. An answer written down with no work will receive no credit.

Problem 1

25 points

Consider an LRC series circuit. When the ac voltage source has a particular frequency ω, the amplitude of the voltage across the inductor is 5 times greater than the peak voltage across the capacitor. Determine ω in terms of the resonant frequency ω_0.

$V_C = \frac{I}{\omega C}$, $V_L = I\omega L$, $\omega_0 = \frac{1}{\sqrt{LC}}$, $V_L = 5V_C$

$$\frac{5I}{\omega C} = I\omega L$$

$$\frac{5}{\omega LC} = \omega$$

$$\frac{1}{LC} = \frac{\omega^2}{5}$$

$$\frac{1}{\sqrt{LC}} = \frac{\omega}{\sqrt{5}}$$

$$\omega_0 = \frac{\omega}{\sqrt{5}}$$

$$\omega = \sqrt{5}\,\omega_0$$

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Problem 2

25 points

You have designed a device that wiggles electric charge around, creating electromagnetic waves. It does so in such a way that the axis of wiggling changes with time, so that the radiated waves are spherically symmetric *on average*. Your device takes 70W of power, and converts that power into "wiggles" with 100% efficiency. Calculate the (average) amplitudes of the electric and magnetic fields 2m away from the device.

Given 100% efficiency, we can say that the wiggler emits a spherically symmetric EM wave with power of 70W.

Surface Area of progagating wave front (sphere): $4\pi r^2$

$$I = \frac{P}{A} = \frac{70}{4\pi(2)^2} = 1.39 \frac{W}{m^2}$$

$$I = S_{avg} = \frac{E_0^2}{2\mu_0 c} \qquad E_0 = \sqrt{2\mu_0 c I}$$

$$= 32.37 \frac{V}{m}$$

$$B_0 = \frac{E_0}{c} = \frac{32.37}{3E8} = 1.08E-7$$

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Problem 3

25 pts

Suppose you manage to produce an electromagnetic wave whose electric field is given by

$$\vec{E} = E_0 \cos\left(\alpha^2 x^2 + \beta^2 t^2 - 2\alpha\beta x t\right)\hat{y},$$

where α, β, E_0 are positive real constants.

$= E_0 \cos\left((\alpha x - \beta t)^2\right)\hat{y}$

(a): 5 pts

In what direction is the wave travelling? (You may use a website like Wolfram Alpha to visualize this wave)

Given fixed phase P_F

$P_F = (\alpha x - \beta t)^2$

if t is increased,

x must also increase,

so wave moves in $+x$ direction

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(b): 20 pts

Derive an expression for the magnetic field of this EM wave.

☐ Direction of propagation is $\vec{E} \times \vec{B}$

(b): 20 pts

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{curl}\left(\langle 0, E_0 \cos((\alpha x - \beta t)^2), 0\rangle\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y}^{0} & \frac{\partial}{\partial z}^{0} \\ E_x^{0} & E_y & E_z^{0} \end{vmatrix} = \frac{\partial}{\partial x} E_y \hat{z}$$

$$\frac{\partial \vec{B}}{\partial t} = -\left(-2\alpha E_0 (\alpha x - \beta t) \sin((\alpha x - \beta t)^2)\right)$$

$$\vec{B} = \int 2\alpha E_0 (\alpha x - \beta t) \sin((\alpha x - \beta t)^2)\, dt$$

$u = \alpha x - \beta t$

$\frac{du}{dt} = -\beta$

$-\frac{du}{\beta} = dt$

$$= -\frac{\alpha E_0}{\beta} \int 2u \sin(u^2)\, du$$

$$= \frac{\alpha E_0}{\beta} \cos(u^2)$$

$$= \frac{\alpha E_0}{\beta} \cos((\alpha x - \beta t)^2)\, \hat{z}$$

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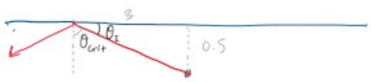
Problem 4

25 pts

Consider a pool of liquid in a room full of air ($n \approx 1$). A beam of light is emitted from a source $h = 0.5$ m below the surface of the liquid, and strikes the surface a distance $d = 3$ m away from the point directly above the source. The beam is totally internally reflected. What can you say about the index of refraction of the liquid?



$$\theta_I = \tan^{-1}\left(\frac{0.5}{3}\right)$$

$$\theta_{crit} = 90 - \theta_I = 90 - \tan^{-1}\left(\frac{0.5}{3}\right) = \sin^{-1}\left(\frac{1}{n}\right)$$

$$\sin\left(90 - \tan^{-1}\left(\frac{0.5}{3}\right)\right) = \frac{1}{n}$$

$$n = \frac{1}{\sin\left(90 - \tan^{-1}\left(\frac{0.5}{3}\right)\right)}$$

index of refraction must be at least 1.0138.
The beam may reflect when shot at a greater angle ($d < 3$), but we don't know.

(although given the lack of significant digits of n(air), we can't really say with certainty that this number is correct. One thing we can say is that $n(\text{liquid}) > n(\text{air})$ for internal reflection to happen at all.