You must show your work to receive credit. An answer written down with no work will receive no credit.

On this entire exam, you may use without proof any expression for impedence and/or phase angle derived in class or the book.

Problem 1

– points

Consider an electromagnetic wave in vacuum whose magnetic field is measured to be

$$\vec{B} = B_0 \cos((\alpha x - \beta t)^2)\hat{z},$$

where α and β are constants such that $\beta/\alpha = c$.

(a)

Find the electric field of this EM wave.

One of Maxwell's differential equations is

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}.$$

We can compute:

$$\begin{aligned}\nabla \times \vec{B} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B_0\cos((\alpha x - \beta t)^2) \end{vmatrix} \\ &= 2B_0\alpha(\alpha x - \beta t)\sin((\alpha x - \beta t)^2)\hat{y}.\end{aligned}$$

One can check that the following electric field satisfies the above equation:

$$\begin{aligned}\vec{E} &= B_0 c^2 \frac{\alpha}{\beta}\cos((\alpha x - \beta t)^2)\hat{y} \\ &= B_0 c \cos((\alpha x - \beta t)^2)\hat{y}\end{aligned}$$

(a)

(b)

Find the intensity of this EM wave, in terms of $B_0, c, \alpha, \beta, \mu_0, x$, and/or ϵ_0. You need not evaluate any integrals you find.

This problem doesn't really make sense if we define the intensity (as we did in class) a the time-average of the intensity over one period, since there is no well-defined period. The clarification on CW was "Instead of computing the intensity in part (b), compute the average power per unit area over some time t_0 carried by the wave."

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{B_0^2 c}{\mu_0}\cos^2((\alpha x - \beta t)^2)\hat{x}.$$

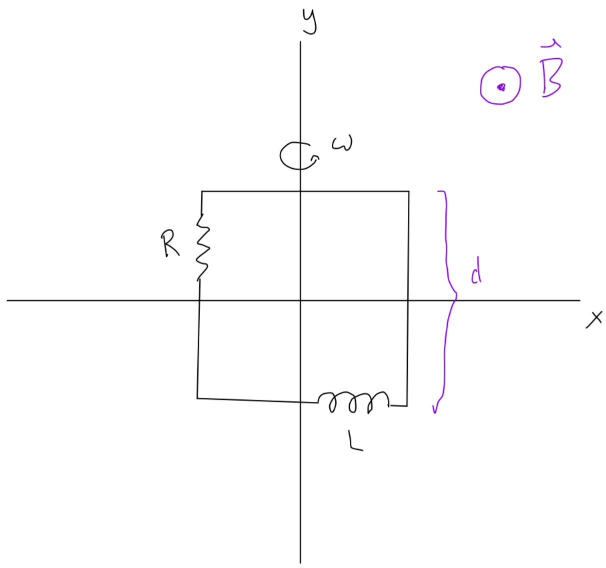
We can then take a time-average of this quantity at time t, over some time period t_0:

$$I_{t_0}(t) = \frac{B_0^2 c}{\mu_0}\left|\frac{1}{t_0}\int_t^{t+t_0}\cos^2((\alpha x - \beta t)^2)dt\right|.$$

Problem 2

– points

Consider a square loop of wire of side length d centered at the origin. Attached in series in the wire are a resistor R and an inductor L. Initially the square lies in the xy plane; the loop begins to rotate with constant speed around the y axis with angular frequency ω. There is a constant, uniform magnetic field $\vec{B} = B\hat{z}$ everywhere.



(a)

Calculate the magnitude (not amplitude!) of the emf in the loop of wire as a function of time.

The magnetic flux through the loop is

$$|\Phi_B(t)| = |Bd^2 \cos \omega t|.$$

By Faraday's law, we have

$$|\mathcal{E}| = |\frac{d\Phi_B}{dt}| = |\omega B d^2 \sin \omega t|.$$

(a)

(b)

Calculate the amplitude of the current induced in the loop.

The loop is driven with a sinusoidal driving emf

$$|v(t)| = |\omega B d^2 \sin \omega t|.$$

The impedance of this circuit is

$$Z(\omega) = \sqrt{R^2 + (\omega L)^2}.$$

Thus the amplitude of the current is

$$I = V/Z(\omega) = \frac{\omega B d^2}{\sqrt{R^2 + (\omega L)^2}}$$

(b)

Problem 3

– points

Consider an LRC series circuit driven by a source supplying the emf

$$v(t) = \alpha \int_0^\infty e^{-\omega/\omega_0} \cos(\omega t) d\omega,$$

where ω_0 and α are constants. Find the current that flows in the circuit, in terms of t, α, ω_0, L, R, and/or C. **Explain your reasoning.** You need not evaluate any integrals you encounter.

This is simply a *continuous* sum of periodic driving emfs weighted by the exponential function. Each "piece" of the current response will additionally be weighted by the impedance:

$$i(t) = \alpha \int_0^\infty \frac{e^{-\omega/\omega_0}}{Z(\omega)} \cos(\omega t - \phi(\omega)) d\omega,$$

where

$$Z(\omega) = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$
$$\phi(\omega) = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right).$$

Note: in a lecture I had incorrectly written the current for multiple oscillating sources *without* the phase factor $\phi(\omega)$. Grading should be lenient with regards to this factor.

Problem 4

– points

Consider an LRC series circuit driven by a source of emf with angular frequency $\omega = 1/\sqrt{LC}$. You measure the amplitude of the charge on the capacitor to be Q_0. Calculate the amplitude of the source emf, as a function of L, R, C, and/or Q.

$$i = \frac{dq}{dt} = Q\omega\cos(\omega t).$$

Then

$$\begin{aligned} V = IZ &= Q\omega Z(\omega) \\ &= Q\omega\sqrt{R^2 + (\omega L - 1/\omega C)^2} \\ &= \frac{QR}{\sqrt{LC}}. \end{aligned}$$