You must show your work to receive credit. An answer written down with no work will receive no credit.

On this entire exam, you may use without proof any expression for impedence and/or phase angle derived in class or the book.

Problem 1

30 points

Consider an electromagnetic wave in vacuum whose magnetic field is measured to be

$$\vec{B} = B_0 \cos((\alpha x - \beta t)^2)\hat{z},$$

where α and β are constants such that $\beta/\alpha = c$. The wavelength of the wave is $\lambda = 2\pi/\alpha$, and its period is $T = 2\pi/\beta$.

(a): 15 pts

Find the electric field of this EM wave.

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \Rightarrow \text{from Maxwell's Equations}$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & B_0\cos[(\alpha x - \beta t)^2] \end{vmatrix} = \frac{\partial}{\partial x}\left[-B_0\cos(\alpha x - \beta t)^2\right]\hat{y}$$

$$\frac{\partial}{\partial x}\left[B_0 \cos\left[(\alpha x - \beta t)^2\right]\right]\hat{y}$$

$$= -B_0 \cdot -\sin(\alpha x - \beta t)^2 \cdot 2(\alpha x - \beta t)\cdot \alpha \; \hat{y}$$

$$= 2B_0\alpha(\alpha x - \beta t)\sin[(\alpha x - \beta t)^2]\hat{y}$$

$$2B_0\alpha(\alpha x - \beta t)\sin[(\alpha x - \beta t)^2]\hat{y} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\therefore \frac{\partial \vec{E}}{\partial t} = 2B_0 c^2 \alpha(\alpha x - \beta t)\sin[(\alpha x - \beta t)^2]\hat{y}$$

$$\vec{E} = \int \frac{\partial \vec{E}}{\partial t}\,dt = 2B_0 c^2 \alpha \int (\alpha x - \beta t) \sin\left[(\alpha x - \beta t)^2\right] \hat{y}$$

$$= 2B_0 c^2 \alpha \cdot \frac{\cos\left[(\beta t - \alpha x)^2\right]}{2\beta} \hat{y}$$

$$= \frac{2B_0 c^2 \alpha}{2\beta} \cos\left[(\beta t - \alpha x)^2\right] \hat{y}$$

$$= B_0 c \cos\left[(\beta t - \alpha x)^2\right] \hat{y}$$

$$\boxed{\vec{E} = B_0 c \cos\left[(\beta t - \alpha x)^2\right] \hat{y}}$$

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(b): 15 pts

Find the intensity of this EM wave, in terms of $B_0, c, \alpha, \beta, \mu_0, x$, and/or ϵ_0. You need not evaluate any integrals you find.

$$S_{av} = \frac{E_0^2}{2\mu_0 c} \qquad E_0 = cB_0$$

$$\therefore S_{av} = \frac{c^2 B_0^2}{2\mu_0 c} = \frac{B_0^2 c}{2\mu_0} \Rightarrow \text{average power per unit area}$$

$$\int_0^{t_0} S_{av}\,dt = \int_0^{t_0} \frac{B_0^2 c}{2\mu_0}\,dt = \frac{B_0^2 c}{2\mu_0} t_0 = \boxed{\frac{B_0^2 c t_0}{2\mu_0}}$$

average power per unit area over a time t_0

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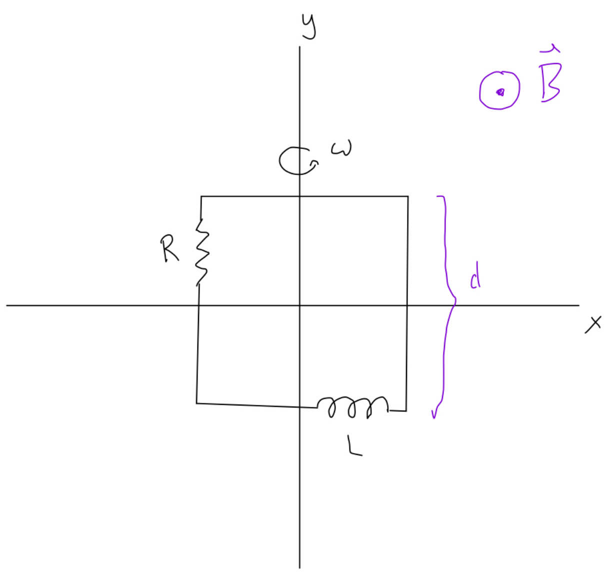
Problem 2

30 points

Consider a square loop of wire of side length d centered at the origin. Attached in series in the wire are a resistor R and an inductor L. Initially the square lies in the xy plane; the loop begins to rotate with constant speed around the y axis with angular frequency ω. There is a constant, uniform magnetic field $\vec{B} = B\hat{z}$ everywhere.



(a): 15 pts

Calculate the magnitude (not amplitude!) of the emf in the loop of wire as a function of time.

Faraday's Law: $\varepsilon = -\frac{d\Phi_B}{dt}$

$$\therefore \text{emf} = -\frac{\nabla\Phi}{\Delta t} = -\frac{\Delta BA\cos\theta}{\Delta t} = -BA\frac{\Delta\cos\omega t}{\Delta t} = -BA\frac{d(\cos\omega t)}{dt}$$

$$= -BA \cdot -\omega\sin\omega t = \omega BA\sin(\omega t) \qquad A = d^2$$

$$\boxed{\varepsilon = \omega B d^2 \sin(\omega t)}$$

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(b): 15 pts

Calculate the amplitude of the current induced in the loop.

current amplitude : max current

$$\mathcal{E} - iR - L\frac{di}{dt} = 0 \quad \Rightarrow \text{Kirchhoff's Loop Rule}$$

$$\frac{di}{dt} = \frac{\mathcal{E} - iR}{L}$$

$$\frac{di}{i - \mathcal{E}/R} = -\frac{R}{L}dt$$

$$\int_0^i \frac{di'}{i' - \mathcal{E}/R} = -\int_0^t \frac{R}{L}dt'$$

$$\ln\left(\frac{i - \mathcal{E}/R}{-\mathcal{E}/R}\right) = -\frac{R}{L}t \Rightarrow i = \frac{\mathcal{E}}{R}\left(1 - e^{-(R/L)t}\right)$$

$$= \frac{\omega B d^2}{R}\left(1 - e^{-(R/L)t}\right)$$

Max current as $t \to \infty$, $i \to \omega B d^2/R$

$$\therefore I = \frac{\mathcal{E}}{R} \qquad \mathcal{E} = \omega B d^2$$

$$\therefore \boxed{I = \frac{\omega B d^2}{R}}$$

Problem 3

25 points

Consider an LRC series circuit driven by a source supplying the emf

$$v(t) = \alpha \int_0^\infty e^{-\omega/\omega_0} \cos(\omega t) d\omega,$$

where ω_0 and α are constants. Find the current that flows in the circuit, in terms of t, α, ω_0, L, R, and/or C. **Explain your reasoning.** You need not evaluate any integrals you encounter.

Use Kirchhoff's Rule:

$$\mathcal{E} - \underbrace{iR}_{\text{resistor}} - \underbrace{L\frac{di}{dt}}_{\text{inductor}} - \underbrace{\frac{q}{C}}_{\text{capacitor}} = 0$$

$$i = \frac{dq}{dt}$$

$$\therefore \quad \mathcal{E} - iR - L\frac{di}{dt} - \frac{q}{C} = 0$$

$$iR + L\frac{di}{dt} + \frac{\int i\,dt}{C} = -\mathcal{E}$$

$$\frac{d}{dt}\left[iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt\right] = -\frac{dv}{dt}$$

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i = -\frac{dv}{dt}$$

$$\boxed{R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i = -\frac{d}{dt}\left[\alpha\int_0^\infty e^{-w/w_0}\cos(wt)\,dw\right]}$$

solve this differential equation to get i

Explanation: Using Kirchhoff's rule, we can write out a second-order differential equation for the general form $v(t) = v_R + v_L + v_C$. Solving this equation will give us $i(t)$, the current that flows in the circuit.

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Problem 4

15 points

resonant frequency

Consider an LRC series circuit driven by a source of emf with angular frequency $\omega = 1/\sqrt{LC}$. You measure the amplitude of the charge on the capacitor to be Q. Calculate the amplitude of the source emf, as a function of L, R, C, and/or Q.

$$V = IZ$$

plug in $\omega = \frac{1}{\sqrt{LC}}$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{R^2 + \left(\frac{\sqrt{L}}{\sqrt{C}} - \frac{\sqrt{LC}}{\sqrt{C}}\right)^2} = \sqrt{R^2} = R$$

$$\therefore V = IR$$

$$\left.\begin{aligned} V_C &= IX_C \\ V_C &= \frac{Q}{C} \end{aligned}\right\} \therefore IX_C = \frac{Q}{C}$$

$$I = \frac{Q}{CX_C} = \frac{Q\omega C}{C} = Q\omega = \frac{Q}{\sqrt{LC}}$$

$$\therefore V = IR = \frac{Q}{\sqrt{LC}} R = \boxed{\frac{QR}{\sqrt{LC}}}$$