You must show your work to receive credit. An answer written down with no work will receive no credit.

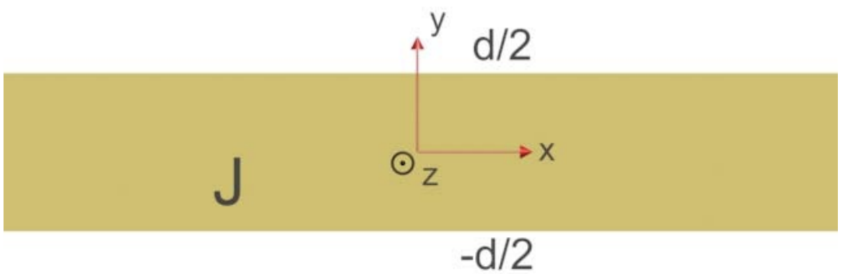
Problem 1

70 points

Consider a volume of current that extends infinitely in the x- and z-directions, and has a thickness of d in the y-direction (henceforth referred to as "the volume"). The current density in the volume is given by

$$\vec{J} = 2J_0 \frac{|y|}{d} \hat{z},$$

where $J_0 > 0$ is a constant with units of current per unit area.



(a): 10 points

Find the direction of the magnetic field in the following regions of space:

1. $y < -d/2$
2. $-d/2 < y < 0$
3. $0 < y < d/2$
4. $y > d/2$

Write your answer in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and/or $\hat{z}$.

The magnetic field is in the $-\hat{x}$ direction for $y > 0$ and in the $+\hat{x}$ direction for $y < 0$.	(a): 10 points

(b): 10 points

Outside of the volume, in which directions could you throw a charged particle if you didn't want it to feel any force from the magnetic field?

You could throw it along the $\pm\hat{x}$ direction, since in those directions $\vec{v} \times \vec{B} = 0$.	(b): 10 points

(c): 5 points

Are there any locations where the magnetic field vanishes? If so, where?

The magnetic field vanishes on the xz-plane, by symmetry.	(c): 5 points

Problem 1 continued on next page...

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(d): 25 points

Use Ampere's law to calculate the magnetic field magnitude everywhere in space (inside and outside the volume). [In terms of d, J_0, μ_0, and/or any spatial coordinates]. You may find your results from parts (a) and (c) useful.

A nice Amperean path is a rectangle whose base (of length L) lies on the x-axis and whose sides (of length y) lie in the $+\hat{y}$ direction. The magnetic field vanishes on the base, and since the two sides are perpendicular to the magnetic field, the only contribution to $\oint \vec{B}\cdot d\vec{l}$ is from the top:

$$\oint \vec{B}\cdot d\vec{l} = BL.$$

By Ampere's law, this is equal to the amount of current enclosed by the Amperean loop:

$$\begin{aligned} BL &= \mu_0 I_{\text{encl}} \\ &= \mu_0 \int_0^L dx \int_0^y dy' |\vec{J}| \\ &= \mu_0 L \begin{cases} \int_0^y dy'(2J_0 \frac{y'}{d}) & 0 < y < d/2 \\ \int_0^{d/2} dy'(2J_0 \frac{y'}{d}) & y > d/2 \end{cases} \\ &= J_0\mu_0 \frac{L}{d} \begin{cases} y^2 & 0 < y < d/2 \\ (d/2)^2 & y > d/2 \end{cases}, \end{aligned}$$

(d): 25 points

so that

$$B = \begin{cases} J_0\mu_0 \frac{y^2}{d} & 0 < y < d/2 \\ \frac{1}{4}J_0\mu_0 d & y > d/2. \end{cases}$$

By symmetry, we can generalize this to

$$B = \begin{cases} J_0\mu_0 \frac{y^2}{d} & |y| < d/2 \\ \frac{1}{4}J_0\mu_0 d & |y| > d/2. \end{cases}$$

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(e): 20 points (5 points each)

Consider a point magnetic dipole $\vec{\mu} = |\vec{\mu}|\hat{y}$.

1. If the dipole is placed *outside* the volume, will it feel a torque? Explain.
2. If the dipole is placed *outside* the volume, will it feel a net force? Explain.
3. If the dipole is placed *inside* the volume (but not at $y = 0$), will it feel a torque? Explain.
4. If the dipole is placed *inside* the volume (but not at $y = 0$), will it feel a net force? Explain.

1. Yes, it will feel a torque that tends to align it with the magnetic field (which is in the $\pm\hat{x}$ direction)
2. No, the field is uniform, and so there will be no net force.
3. Yes, it will feel a torque that tends to align it with the magnetic field (which is in the $\pm\hat{x}$ direction)
4. This was a little tricky. The net force on the magnetic dipole inside the slab superficially appears to be zero, since the direction of the dipole is perpendicular to the direction of the magnetic field. However, the dipole will feel a torque rotating towards the direction of the magnetic field, after which it will indeed feel a net force due to the inhomogeneity of the magnetic field.

(e): 20 points

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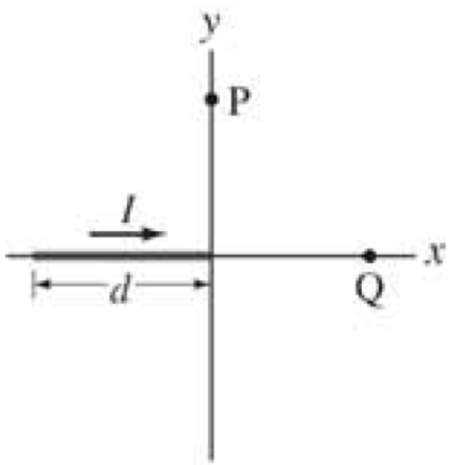
Problem 2

30 points

Consider a segment of wire of length d carrying current I along the $+\hat{x}$ direction. In this problem, you may find the integrals found on this page useful:
Hyperphysics table of integrals



(a): 10 points

Use the Biot-Savart law to calculate the magnetic field at point Q (a distance x along the positive x axis).

The Biot-Savart law is

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2}.$$

Here $d\vec{l} = \hat{x} dx$, and $\hat{r} = \hat{x}$, so $d\vec{l} \times \hat{r} = 0$ for the entire integrand. Thus

$$\vec{B}(x) = 0.$$

(a): 10 points

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(b): 20 points

Use the Biot-Savart law to calculate the magnetic field at point P (a distance y along the positive y axis).

The Biot-Savart law is

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I d\vec{l} \times \hat{r}}{r^2}.$$

Here $d\vec{l} = \hat{x}dx$. $d\vec{l} \times \hat{r}$ will always point in the $+\hat{z}$ direction, so

$$\begin{aligned}\vec{B}(y) &= \frac{\mu_0 I}{4\pi}\int_{-d}^{0} dx \frac{|\hat{x}\times\hat{r}|}{r^2}\hat{z} \\ &= \frac{\mu_0 I}{4\pi}\int_{-d}^{0} dx \frac{\sin\theta}{r^2}\hat{z},\end{aligned}$$

where θ is the angle between $\hat{x}$ and $\hat{r}$. θ is related to the coordinates as $\sin\theta = \frac{y}{r}$, so that

$$\vec{B}(y) = \frac{\mu_0 I}{4\pi} y \int_{-d}^{0} dx \frac{1}{r^3}\hat{z}.$$

(b): 20 points

Finally, $r = \sqrt{x^2+y^2}$, so

$$\vec{B}(y) = \frac{\mu_0 I}{4\pi} y \int_{-d}^{0} dx \frac{1}{(x^2+y^2)^{3/2}}\hat{z}.$$

Consulting the given table of integrals, we find

$$\begin{aligned}\vec{B}(y) &= \frac{\mu_0 I}{4\pi} y \left[\frac{x}{y^2\sqrt{x^2+y^2}}\right]_{-d}^{0}\hat{z} \\ &= \frac{\mu_0 I}{4\pi}\frac{d}{y\sqrt{d^2+y^2}}\hat{z}\end{aligned}$$