You must show your work to receive credit. An answer written down with no work will receive no credit.

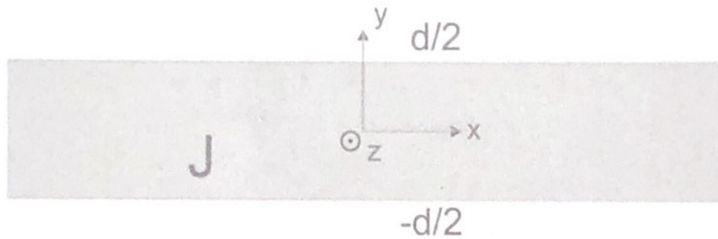
Problem 1

70 points

Consider a volume of current that extends infinitely in the x- and z-directions, and has a thickness of d in the y-direction (henceforth referred to as "the volume"). The current density in the volume is given by

$$\vec{J} = 2J_0 \frac{|y|}{d} \hat{z},$$

where $J_0 > 0$ is a constant with units of current per unit area.



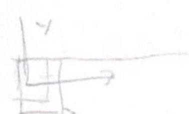
(a): 10 points

Find the direction of the magnetic field in the following regions of space:

1. $y < -d/2$
2. $-d/2 < y < 0$
3. $0 < y < d/2$
4. $y > d/2$

right hand rule

Write your answer in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and/or $\hat{z}$.



1, $y < -\frac{d}{2}$ $\hat{x}$ direction

2. $-\frac{d}{2} < y < 0$ $\hat{x}$ direction

3, $0 < y < d/2$ $-\hat{x}$ direction

4. $y < \frac{d}{2}$ $-\hat{x}$ direction

Problem 1 continued on next page...

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(b): 10 points

Outside of the volume, in which directions could you throw a charged particle if you didn't want it to feel any force from the magnetic field?

Because the force on a charged particle is $\vec{F} = q\vec{v} \times \hat{B}$,

You should throw the particle in the $\pm\hat{x}$ directions.

Because $\hat{B}$ is in the $\hat{x}$ direction when $y < -\frac{d}{2}$, and is in the $-\hat{x}$ direction when $y > \frac{d}{2}$, throwing the particle in the $\pm\hat{x}$ directions will cause $\vec{v} \times \vec{B}$ to evaluate to 0, so the particle will feel 0 magnetic force.

(c): 5 points

Are there any locations where the magnetic field vanishes? If so, where?

The magnetic field vanishes at $y = 0$ because of symmetry around it. At $y = 0$, the magnetic field produced by the current above $y = 0$ cancels with the magnetic field produced by the current below.

Problem 1 continued on next page...

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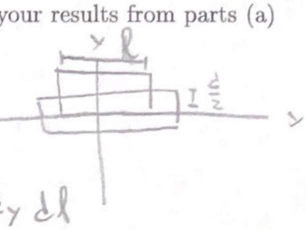
(d): 25 points

Use Ampere's law to calculate the magnetic field magnitude everywhere in space (inside and outside the volume). [In terms of d, J_0, μ_0, and/or any spatial coordinates]. You may find your results from parts (a) and (c) useful.

1. $\boxed{0 < y < \frac{d}{2}}$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$ $\qquad I = \int J \, dA \qquad A = ly \qquad dA = dl\,dy$



$B \oint dl = \mu_0 I_{encl} \qquad I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^y 2 J_0 \frac{|y|}{d} \hat{z} \, dy \, dl$

$B l = -\frac{\mu_0 J_0 l y^2}{d} \hat{x} \qquad I = \frac{2 J_0 l y^2}{d} \hat{z}$

$B(y) = -\frac{\mu_0 J_0 y^2}{d} \hat{x}$

2. $\boxed{-\frac{d}{2} < y < 0}$, same as 1., except the magnetic field is in opposite direction.

$B(-y) = \frac{\mu_0 J_0 y^2}{d} \hat{x}$

3. $\boxed{y > \frac{d}{2}}$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} \qquad I = \int J \, dA$

$B \oint dl = \mu_0 I_{encl} \qquad I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{\frac{d}{2}} 2 J_0 \frac{|y|}{d} \hat{z} \, dy \, dl$

$I = \cancel{2} J_0 \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{y^2}{\cancel{2} d} \Big|_0^{\frac{d}{2}} dl$

$B l = \frac{\mu_0 J_0 l d}{4} \qquad = J_0 \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{d}{4}$

$B = -\frac{\mu_0 J_0 d}{4} \hat{x} \qquad = \frac{J_0 l d}{4}$

4. $\boxed{y < -\frac{d}{2}}$, same as 3., except the magnetic field is in opposite direction

$B = \frac{\mu_0 J_0 d}{4} \hat{x}$

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(e): 20 points (5 points each)

Consider a point magnetic dipole $\vec{\mu} = |\vec{\mu}|\hat{y}$.

1. If the dipole is placed *outside* the volume, will it feel a torque? Explain.
2. If the dipole is placed *outside* the volume, will it feel a net force? Explain.
3. If the dipole is placed *inside* the volume (but not at $y = 0$), will it feel a torque? Explain.
4. If the dipole is placed *inside* the volume (but not at $y = 0$), will it feel a net force? Explain.

1. The dipole will feel a net torque. Outside the volume, the magnetic field is uniform, so the magnetic torque can be found through $\tau = \vec{\mu} \times \vec{B}$. Because $\vec{\mu}$ is in the $\hat{y}$ direction, and $\vec{B}$ is in the $\hat{x}$ direction, there will be torque. 3. The same is true inside the volume, but because the magnetic field is not uniform, $\tau = \vec{\mu} \times \vec{B}$ is only an approximation.

2. The dipole will not feel a net force. Since $U_B = -\vec{\mu} \cdot \vec{B}$, and $\vec{\mu}$ is in the $\hat{y}$ and $\vec{B}$ is in the $\hat{x}$ direction, the dot product will be 0.

4. The dipole will not feel a net force because the magnetic field and the point magnetic dipole $\vec{\mu}$ are in perpedicular directions, so the dot product $U_B = -\vec{\mu} \cdot \vec{B}$ evaluates to 0.

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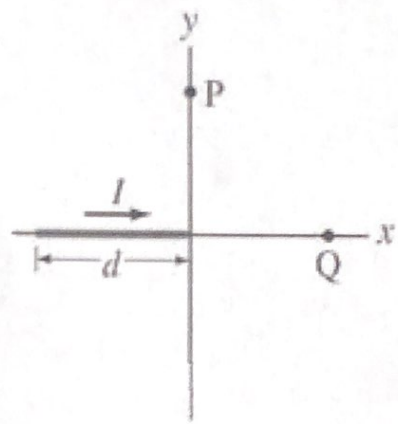
Problem 2

30 points

Consider a segment of wire of length d carrying current I along the $+\hat{x}$ direction. In this problem, you may find the integrals found on this page useful:
Hyperphysics table of integrals



(a): 10 points

Use the Biot-Savart law to calculate the magnetic field at point Q (a distance x along the positive x axis).

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^2} \qquad d\vec{l} = dx\, \hat{x}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \hat{r}}{r^2}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, dx}{r^2}\, \hat{x} \times \hat{x}$$

r points in $\hat{x}$ direction

$\vec{B} = 0$, because $d\vec{l}$ and $\hat{r}$ both point in $\hat{x}$ direction.

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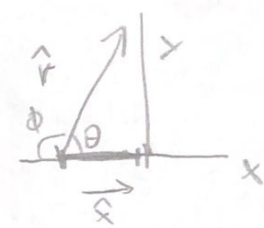
(b): 20 points

Use the Biot-Savart law to calculate the magnetic field at point P (a distance y along the positive y axis).

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \hat{r}}{r^2} \qquad dl = dx\,\hat{x} \qquad r = \sqrt{x^2+y^2}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-d}^{0} \frac{dx}{x^2+y^2}\, \hat{x} \times \hat{r}$$



$$\vec{B}(y) = \frac{\mu_0 I y}{4\pi} \int_{-d}^{0} \frac{dx}{(x^2+y^2)^{\frac{3}{2}}}\, \hat{z}$$

$$\hat{x} \times \hat{r} = -\sin\phi\, \hat{z}$$

$$= \sin(\pi - \phi)\hat{z}$$

$$= \frac{y}{r}\hat{z}$$

$$= \frac{\mu_0 I y}{4\pi} \left(\frac{x}{y^2\sqrt{x^2+y^2}} \right) \Bigg|_{-d}^{0}$$

$$= \frac{\mu_0 I d}{4\pi y \sqrt{d^2+y^2}}\, \hat{z}$$