You must show your work to receive credit. An answer written down with no work will receive no credit.

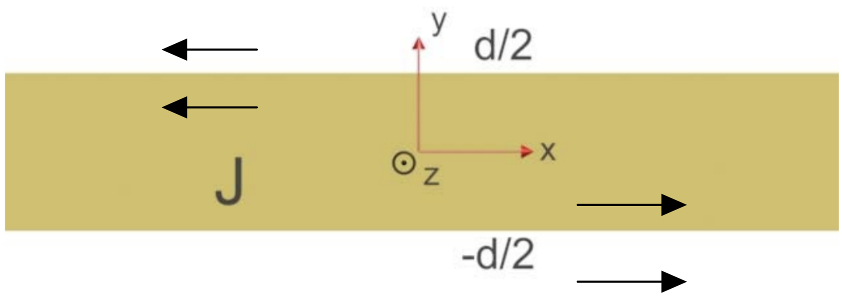
Problem 1

70 points

Consider a volume of current that extends infinitely in the x- and z-directions, and has a thickness of d in the y-direction (henceforth referred to as "the volume"). The current density in the volume is given by

$$\vec{J} = 2J_0 \frac{|y|}{d} \hat{z},$$

where $J_0 > 0$ is a constant with units of current per unit area.



(a): 10 points

Find the direction of the magnetic field in the following regions of space:

1. $y < -d/2$ $\hat{x}$
2. $-d/2 < y < 0$ $\hat{x}$
3. $0 < y < d/2$ $-\hat{x}$
4. $y > d/2$ $-\hat{x}$

Write your answer in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and/or $\hat{z}$.

1. Using right hand rule, $\vec{B}$ in $\hat{x}$ sums up. Other components cancel
2. same reason as (1)
3. Using right hand rule, $\vec{B}$ in $-\hat{x}$ sums up. Other components cancel
4. same reason as (3)

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(b): 10 points

Outside of the volume, in which directions could you throw a charged particle if you didn't want it to feel any force from the magnetic field?

You could throw the charge particle in either $\hat{x}$ or $-\hat{x}$ direction. $F = q\vec{v}\times\vec{B}$. If $\vec{v}$ and $\vec{B}$ are parallel to each other, $\vec{v}\times\vec{B}$ is 0. Therefore, the force it experiences is 0.

(c): 5 points

Are there any locations where the magnetic field vanishes? If so, where?

Yes, at $y=0$. $\vec{J}$ is 0 at this point, meaning that there's no current to create a magnetic field.

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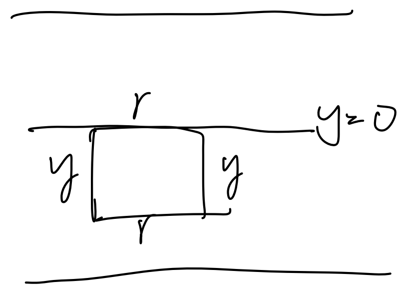
(d): 25 points

Use Ampere's law to calculate the magnetic field magnitude everywhere in space (inside and outside the volume). [In terms of d, J_0, μ_0, and/or any spatial coordinates]. You may find your results from parts (a) and (c) useful.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$



$$B \cdot r = \mu_0 \cdot \left(\int_0^y J\,dy \cdot r \right)$$

$$B = \mu_0 \int_0^y J\,dy$$

$$= \mu_0 \int_0^y 2J_0 \frac{|y|}{d}\,dy$$

$$= \frac{2\mu_0 J_0}{d} \left. \frac{y^2}{2} \right|_0^y$$

$$= \frac{\mu_0 J_0 y^2}{d}$$

$y \leq -d/2$: $\vec{B} = \frac{\mu_0 J_0 \left(\frac{d}{2}\right)^2}{d} = \frac{\mu_0 J_0 d}{4}\hat{x}$, $|\vec{B}| = \frac{\mu_0 J_0 d}{4}$

$-d/2 < y < 0$: $\vec{B} = \frac{\mu_0 J_0 y^2}{d}\hat{x}$, $|\vec{B}| = \frac{\mu_0 J_0 y^2}{d}$

$y = 0$: $\vec{B} = 0$, $|\vec{B}| = 0$

$0 < y < \frac{d}{2}$: $\vec{B} = \frac{-\mu_0 J_0 y^2}{d}\hat{x}$, $|\vec{B}| = \frac{\mu_0 J_0 y^2}{d}$

$y \geq \frac{d}{2}$: $\vec{B} = \frac{-\mu_0 J_0 d}{4}\hat{x}$, $|\vec{B}| = \frac{\mu_0 J_0 d}{4}$

Problem 1 continued on next page...

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(e): 20 points (5 points each)

Consider a point magnetic dipole $\vec{\mu} = |\vec{\mu}|\hat{y}$.

1. If the dipole is placed *outside* the volume, will it feel a torque? Explain.
2. If the dipole is placed *outside* the volume, will it feel a net force? Explain.
3. If the dipole is placed *inside* the volume (but not at $y = 0$), will it feel a torque? Explain.
4. If the dipole is placed *inside* the volume (but not at $y = 0$), will it feel a net force? Explain.

1. Yes. $\vec{\tau} = \vec{\mu} \times \vec{B}$. Since $\vec{\mu}$ (with direction $\hat{y}$) and $\vec{B}$ (with direction $\hat{x}$ or $-\hat{x}$) are perpendicular, their cross product exists, exerting a non-zero torque on the dipole.

2. No. Since $\vec{B}$ outside of the volume is relatively constant, the dipole will experience equal and opposite forces on its two ends, therefore, it will not experience a net force.

3. Yes. $\vec{\tau} = \vec{\mu} \times \vec{B}$. Since $\vec{\mu}$ (with direction $\hat{y}$) and $\vec{B}$ (with direction $\hat{x}$ or $-\hat{x}$) are perpendicular, their cross product exists, exerting a non-zero torque on the dipole.

4. Yes, Since $\vec{B}$ inside the volume varies in y-direction, the dipole will not experience equal and opposite forces when its in y-direction. Therefore, it will experience a net force

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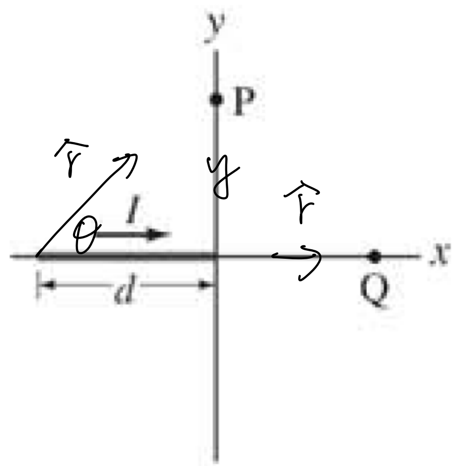
Problem 2

30 points

Consider a segment of wire of length d carrying current I along the $+\hat{x}$ direction. In this problem, you may find the integrals found on this page useful:
Hyperphysics table of integrals



$$\sin\theta = \frac{y}{\sqrt{l^2+y^2}}$$

(a): 10 points

Use the Biot-Savart law to calculate the magnetic field at point Q (a distance x along the positive x axis).

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times \hat{r}}{r^2}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_0^d \frac{I d\vec{l} \times \hat{r}}{(l+x)^2}$$

$\vec{l} \times \hat{r} = 0$ since they are parallel

$\vec{B}$ at $Q = 0$

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(b): 20 points

Use the Biot-Savart law to calculate the magnetic field at point P (a distance y along the positive y axis).

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I \sin\theta \, dl}{l^2 + y^2} \hat{z} = \frac{\mu_0}{4\pi} \frac{I}{l^2+y^2} \frac{y}{\sqrt{l^2+y^2}} dl \, \hat{z}$$

$$\vec{B} = \frac{\mu_0 I y}{4\pi} \int_0^d \frac{1}{(l^2+y^2)^{3/2}} dl \, \hat{z} = \frac{\mu_0 I y}{4\pi} \frac{d}{y^2 \sqrt{d^2+y^2}} \hat{z} = \frac{\mu_0 I d}{4\pi y \sqrt{d^2+y^2}} \hat{z}$$