Physics 1C Midterm

TOTAL POINTS

89 / 100

QUESTION 1

Problem 1 40 pts

1.1 (a) **12 / 15**

- **0 pts** Correct
- **3 pts** missing/incorrect magnetic field directions: between wires into the page, outside wires out of the page

- 3 Point adjustment

incorrect direction in between wires (should be (-z) direction), simplification errors with distances

1.2 (b) **15 / 15**

✓ **- 0 pts Correct**
- **3 pts** error in integration bounds
- **3 pts** incorrect initial setup

1.3 (c) **10 / 10**

✓ **- 0 pts Correct**
- **4 pts** incorrect use of Faraday's Law
- **4 pts** incorrect use of Ohm's Law
- **2 pts** incorrect/missing current direction

QUESTION 2

Problem 2 40 pts

2.1 (a) **15 / 15**

✓ **- 0 pts Correct**

2.2 (b) **10 / 15**

- **0 pts** Correct

✓ **- 5 pts negative frequency**
- **8 pts** incorrect determination of omega_+-
- **5 pts** imaginary frequency
- **10 pts** No determination of omega_+-
- **5 pts** Solved quadratic equation incorrectly
- **3 pts** Impedance should be doubled
- **5 pts** No resonance frequency

2.3 (c) **7 / 10**

- **0 pts** Correct
- **3 pts** Changing C doesn't change FWHM
- **5 pts** Incorrect reasoning

✓ **- 0 pts ok given wrong answer for (b)**

✓ **- 3 pts Need to keep resonant frequency constant!**

QUESTION 3

3 Problem 3 **20 / 20**

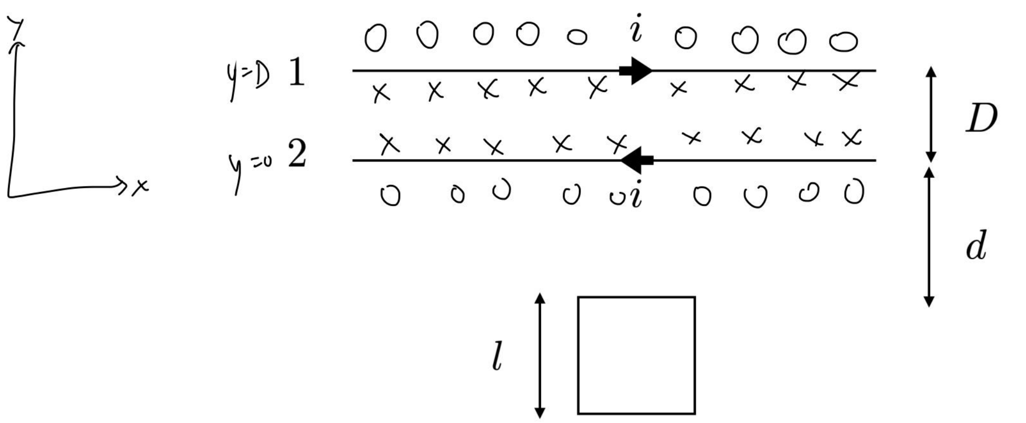
- **5 pts** Did not verify Gauss' law for magnetic fields.
- **5 pts** Did not verify Faraday's law.
- **5 pts** Did not verify Ampere's law.
- **5 pts** Did not verify Gauss' law for electric fields.

✓ **- 0 pts Correct**
- **2.5 pts** Error verifying Ampere's law.
- **2.5 pts** Error verifying Gauss' law for electric fields.
- **2.5 pts** Error verifying Gauss' law for magnetic fields.
- **2.5 pts** Error verifying Faraday's law.

You must show your work to receive credit. An answer written down with no work will receive no credit.

Problem 1

Two infinite parallel wires separated by distance D carry equal currents $i(t)$ in opposite directions. These currents depend on time, but are always equal in magnitude and in opposite directions. A square loop of wire with side length l lies in the plane of the wires, its closest edge a distance d from the closest wire. The square loop has total resistance R.



(a): 15 pts

Derive an expression for the magnetic field on the plane containing the wires and the loop. Your answer may or may not depend on i, d, D, l, and R. Remember a magnetic field has both a magnitude and direction.

Givens:

$\vec{B}_{net} = \vec{B}_1 + \vec{B}_2$

Biot-Savart $B(r) = \frac{\mu_0 I}{2\pi r}$ distance from observation point

$\vec{B}_{net}$ below wire 2:

$$\vec{B}_2 = \frac{\mu_0 i(t)}{2\pi(-y)}(\hat{z})$$

$$\vec{B}_1 = \frac{\mu_0 i(t)}{2\pi(D-y)}(-\hat{z})$$

$$\vec{B}_{net} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 i(t)}{2\pi(-y)} - \frac{\mu_0 i(t)}{2\pi(D-y)}$$

$$\boxed{\vec{B}_{net} = \frac{-\mu_0 i(t)}{2\pi} \cdot \frac{D}{Dy-y^2}\hat{z}}$$

$\vec{B}_{net}$ above wire 1:

$$\vec{B}_1 = \frac{\mu_0 i(t)}{2\pi(y-D)}\hat{z}$$

$$\vec{B}_2 = \frac{\mu_0 i(t)}{2\pi(y)}(-\hat{z})$$

$$\boxed{\vec{B}_{net} = \frac{-\mu_0 i(t)}{2\pi} \cdot \frac{D}{Dy-y^2}\hat{z}}$$

$\vec{B}_{net}$ inbetween wire 1 and 2

$$\vec{B}_1 = \frac{\mu_0 i(t)}{2\pi(D-y)}(-\hat{z})$$

$$\vec{B}_2 = \frac{\mu_0 i(t)}{2\pi(y)}(-\hat{z})$$

$$\boxed{\vec{B}_{net} = \frac{-\mu_0 i(t)}{2\pi} \cdot \frac{D}{Dy-y^2}\hat{z}}$$

Thus, the magnetic field is $\boxed{\vec{B}(t,y) \ \frac{-\mu_0 i(t)}{2\pi} \cdot \frac{D}{Dy-y^2}\hat{z}}$

1.1 (a) **12 / 15**

- **- 0 pts** Correct
- **- 3 pts** missing/incorrect magnetic field directions: between wires into the page, outside wires out of the page

- 3 Point adjustment

incorrect direction in between wires (should be (-z) direction), simplification errors with distances

You must show your work to receive credit. An answer written down with no work will receive no credit.

(b): 15 pts

What is the total magnetic flux through the square loop at time t? Your answer may or may not depend on i, d, D, l, and R. [If you could not solve part (a), just use a placeholder $\vec{B}(t, x, y, z)$ for the magnetic field.]

Given

axes are

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A} \qquad dA = l \cdot dy$$

$$\vec{B} = -\frac{\mu_0 i(t)}{2\pi} \frac{D}{Dy - y^2} \hat{z}$$

Bounds of integration: $b = -d$, $a = -(d+l)$ } these are due to the fact that I designated the loop as below the y-axis

$$\Phi_B = \int_a^b B \cdot l\,dy$$

$$= \frac{\mu_0 i(t) l}{2\pi} \int_a^b \frac{D}{Dy - y^2}\, dy$$

$$= \frac{\mu_0 i(t) l}{2\pi} \int_a^b \frac{1}{D-y} + \frac{1}{y}\, dy$$

$$= \frac{\mu_0 i(t) l}{2\pi} \left[\int_a^b \frac{1}{D-y}\, dy + \int_a^b \frac{1}{y}\, dy \right]$$

$$= \frac{l \mu_0 i(t)}{2\pi} \Big[-\ln(y - D) + \ln(y) \Big]_a^b$$

$$= \frac{l \mu_0 i(t)}{2\pi} \Big[-\ln(b-D) + \ln(a-D) + \ln(b) - \ln(a) \Big]$$

$$= \frac{l \mu_0 i(t)}{2\pi} \left[\ln\left(\frac{a-D}{b-D}\right) + \ln\left(\frac{b}{a}\right) \right]$$

$$= \frac{l \mu_0 i(t)}{2\pi} \left[\ln\left(\frac{a-D}{b-D} \cdot \frac{b}{a}\right) \right]$$

$$= \frac{l \mu_0 i(t)}{2\pi} \left[\ln\left(\frac{(-d-l-D)(-d)}{(-d-D)(-d-l)}\right) \right]$$

$$\boxed{\Phi_B = \frac{\mu_0 i(t) l}{2\pi} \left[\ln\left(\frac{(d+l+D)(d)}{(d+D)(d+l)}\right) \right]}$$

let $\frac{(d+l+D)(d)}{(d+D)(d+l)} = m$

for the next question to simplify writing it down

1.2 (b) **15 / 15**

- ✓ **- 0 pts** Correct
- **- 3 pts** error in integration bounds
- **- 3 pts** incorrect initial setup

You must show your work to receive credit. An answer written down with no work will receive no credit.

(c): 10 pts

Find the magnitude and direction of the current induced in the square loop. [If you could not solve part (a), just use a placeholder $\Phi_B(t)$ for the magnetic flux.]

Givens:

$$\Phi_B(t) = \frac{\mu_0 i(t) l}{2\pi} \ln(m)$$

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$I = \frac{\mathcal{E}}{R}$$

$$\mathcal{E} = \frac{-d}{dt} \frac{\mu_0 i(t) l}{2\pi} \ln(m)$$

$$\mathcal{E} = \frac{\mu_0 l}{2\pi} \ln(m) \frac{di}{dt}$$

$$I = \frac{\mathcal{E}}{R}$$

$$\boxed{I_{induced} = \frac{-\mu_0 l}{2\pi R} \ln(m) \frac{di}{dt}}$$

Direction of Induced current:

Since the induced current depends on the function $\frac{di}{dt}$, we see that the direction of the induced current will depend on the sign of $i(t)$, since $i(t)$ can take both positive and negative values. Thus, the induced current will be counter clockwise in these two conditions: if $i(t)$ and the flux are either positive or negative AND $\frac{di}{dt}$ and $\frac{d\Phi_B}{dt}$ are greater than $\emptyset$

Otherwise the induced current will be counter clockwise if $i(t)/\Phi_B$ are either positive or negative but $\frac{di}{dt} / \frac{d\Phi_B}{dt}$ are less than $\emptyset$.

1.3 (C) **10 / 10**

✓ **- 0 pts** **Correct**

- 4 pts incorrect use of Faraday's Law

- 4 pts incorrect use of Ohm's Law

- 2 pts incorrect/missing current direction

You must show your work to receive credit. An answer written down with no work will receive no credit.

Problem 2

You'd like to design a door that only opens when you whistle a particular note. You connect the door latch to a circuit element with resistance R. You have another element that converts a sound wave frequency f to an oscillating voltage of amplitude V_0 and angular frequency $\omega = 2\pi f$. You connect an inductor L and a capacitor C to the resistor R, in order to create an LRC series circuit. The resistor R will "activate" and unlock the door when at least half of the maximum current runs through it.

(a): 15 pts

What is the impedance Z of this circuit, in terms of R, L, C, and ω?

Given that this is an LRC circuit,
we can use Kirchoff's Loop rule and
these values: $v_R = IR\cos\omega t$

$v_L = IX_L\cos(\omega t + \frac{\pi}{2})$

$v_C = IX_C\cos(\omega t - \frac{\pi}{2})$

to find the impedence, which is defined by the equation $V = IZ$

From Pythagorean Theorem

$$V = \sqrt{v_R^2 + (V_L - V_C)^2}$$

$$= \sqrt{(IR)^2 + (IX_L - IX_C)^2}, \text{ where } X_L = \omega L, \; X_C = \frac{1}{\omega C}$$

$$= \sqrt{I^2(R^2 + (X_L - X_C)^2}$$

$$V = I\sqrt{R^2 + (X_L - X_C)^2}$$

thus, $\boxed{Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$

2.1 (a) **15 / 15**

✓ **- 0 pts** Correct

You must show your work to receive credit. An answer written down with no work will receive no credit.

(b): 15 pts

At what frequency ω_0 will the maximum current flow through the circuit? At what frequencies $\omega_\pm$ will *half* the maximum current flow through the circuit? The quantity $\Delta\omega = \omega_+ - \omega_-$ is the *full width at half maximum* (FWHM) of this current. Express your answers in terms of R, L, and C.

Given that the current for an LRC circuit is defined as

$I = \frac{V}{Z}$, or $I = \frac{V}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$

I_{max} occurs when $I = \frac{V}{R}$, or when $(\omega L - \frac{1}{\omega C})^2 = 0$

thus we can solve for ω_0

$(\omega_0 L - \frac{1}{\omega_0 C})^2 = \varnothing$

$\omega_0 \cdot L = \frac{1}{\omega_0 C}$

$\omega_0^2 = \frac{1}{LC}$, thus $\boxed{\omega_0 = \sqrt{\frac{1}{LC}}}$

$$\frac{V_0}{2R} = \frac{V_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$\Rightarrow 2R = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$

$\Rightarrow 4R^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$

$\Rightarrow 3R^2 = (\omega L - \frac{1}{\omega C})^2$

$\Rightarrow \sqrt{3R^2} = \omega L - \frac{1}{\omega C}$

$\Rightarrow \sqrt{3R^2} + \frac{1}{\omega C} = \omega L$

$\Rightarrow \omega\sqrt{3R^2} + \frac{1}{C} = \omega^2 L$

$\omega^2 L - \omega\sqrt{3R^2} - \frac{1}{C} = \varnothing$

$$\boxed{\omega_\pm = \frac{\sqrt{3R^2} \pm \sqrt{\frac{4L}{C} + 3R^2}}{2L}}$$

$$\Delta\omega = \frac{\sqrt{3R^2} + \sqrt{\frac{4L}{C} + 3R^2} - \sqrt{3R^2} + \sqrt{\frac{4L}{C} + 3R^2}}{2L}$$

$$\boxed{\Delta\omega = \frac{\sqrt{\frac{4L}{C} + 3R^2}}{L}}$$

Problem 2 continued on next page...

2.2 (b) **10 / 15**

- **- 0 pts** Correct
- ✓ **- 5 pts negative frequency**
- **- 8 pts** incorrect determination of omega_+-
- **- 5 pts** imaginary frequency
- **- 10 pts** No determination of omega_+-
- **- 5 pts** Solved quadratic equation incorrectly
- **- 3 pts** Impedance should be doubled
- **- 5 pts** No resonance frequency

You must show your work to receive credit. An answer written down with no work will receive no credit.

(c): 10 pts

You'd like your door to open for a small range of frequencies around middle C, approx. 260 Hz. You test out your door and find that it opens for frequencies in the range 210 Hz - 310 Hz; not very secure. Describe what happens to the FWHM $\Delta\omega$ of the current as you vary each of L, R, and C, while keeping the other two fixed. No numerical answers are required. What **qualitative** changes could you make to your circuit to tighten the range of opening frequencies to 240 Hz - 280 Hz?

Given that FWHM $\Delta\omega$ is defined as:

$$\Delta\omega = \frac{\sqrt{\frac{4L}{C} + 3R^2}}{L}$$

if we were to increase:

L: the frequency would go down as the numerator "L" goes up at $\sqrt{L}$ while the denominator "L" goes up at L, thus the denominator would grow faster, decreasing FWHM

R: The frequency would increase as the numerator would grow

C: The frequency would decrease since the numerator would be affected by the $\frac{1}{C}$ factor in one of the terms

if we were to decrease

L: frequency would increase

R: frequency would decrease

C: frequency would increase

(They would all have the opposite effect as when increasing these factors)

Qualitative Changes:

We could do the following things to tighten the range, We could first start by decreasing R since it contributes the most to increasing the frequency. After testing it again, if we were to find that the frequency range dropped to around 280 Hz, we could stop there and then decrease L since that is the biggest factor in dropping the frequency, hopefully thus tightening up the lower end. Then, depending on the range that was left, if we slowly decreased C to incrementally increase the frequency and slowly increase L, to incrementally drop the frequency, it is possible that the range of opening frequencies would be between 240hz to 280 Hz.

2.3 (c) **7 / 10**

- **- 0 pts** Correct
- **- 3 pts** Changing C doesn't change FWHM
- **- 5 pts** Incorrect reasoning
- ✓ **- 0 pts ok given wrong answer for (b)**
- ✓ **- 3 pts Need to keep resonant frequency constant!**

You must show your work to receive credit. An answer written down with no work will receive no credit.

Problem 3

20 pts

In class, we have seen plane wave solutions to Maxwell's equations of the form

$$\vec{E} = E_0 \cos(\omega t - kx)\hat{y}.$$

At a given time, such a wave repeats indefinitely along the x-axis; it isn't well-localized in space. EM waves that are more localized in space are described by *wavepackets*, superpositions of plane waves travelling in the same direction, but with many different wavelengths. A general wavepacket for the electric and magnetic fields of an EM wave polarized in the $\hat{y}$-direction travelling along the x direction can be written as

$$\vec{E} = E_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx)\,\hat{y}$$
$$\vec{B} = B_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx)\,\hat{z}$$

where $f(k)$ is a function specifying how "much" of the wavepacket consists of waves of wavenumber k.

Verify that $\vec{E}$ and $\vec{B}$ satisfy the four differential Maxwell equations in empty space:

1) $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

2) $\nabla \times \vec{B} = \mu_0\epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

3) $\nabla \cdot \vec{E} = 0$

4) $\nabla \cdot \vec{B} = 0.$

Givens:

1) $B_0 = \dfrac{E_0}{c}$

2) $\dfrac{c}{\omega} = \dfrac{1}{k}$

3) $c^2 = \dfrac{1}{\mu_0 \varepsilon_0}$

1) Given $\vec{E} = E_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx)\hat{y}$

$\nabla \times \vec{E} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle E_x, E_y, E_z \rangle$

No component of E in $\hat{x}$ and $\hat{z}$

$= \left(\cancel{\frac{\partial E_z}{\partial y}}^{\,0} - \frac{\partial E_y}{\partial z}\right)\hat{x} - \left(\cancel{\frac{\partial E_z}{\partial x}}^{\,0} - \cancel{\frac{\partial E_x}{\partial z}}^{\,0}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}}^{\,0}\right)\hat{z}$

$= -\cancel{\frac{\partial E_y}{\partial z}}^{\,0}\hat{x} + \frac{\partial E_y}{\partial x}\hat{z}$

no z variable in Ey

$= \frac{\partial}{\partial x} E_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx)\,\hat{z}$

let $E_0 = B_0 \cdot c$

$= -k E_0 \int_{-\infty}^{\infty} f(k)\sin(\omega t - kx)\,\hat{z}$

let $k = \frac{\omega}{c}$

$= -k B_0 \cdot c$

$= -\omega B_0 \int_{-\infty}^{\infty} f(k)\sin(\omega t - kx)\,\hat{z}$

$= \frac{\partial}{\partial t}\left(B_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx)\,\hat{z}\right)$

$= \frac{\partial}{\partial t} \cdot \vec{B}$

thus $\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

2) $\nabla \times \vec{B} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \left\langle B_x, B_y, B_z \right\rangle$

$$= \left(\frac{\partial B_z}{\partial y} - \cancel{\frac{\partial B_y}{\partial z}}^{\,0} \right)\hat{x} - \left(\frac{\partial B_z}{\partial x} - \cancel{\frac{\partial B_x}{\partial z}}^{\,0} \right)\hat{y} + \left(\cancel{\frac{\partial B_y}{\partial x}}^{\,0} - \cancel{\frac{\partial B_x}{\partial y}}^{\,0} \right)\hat{z}$$

→ no component of $\vec{B}$ in $\hat{x}$ and $\hat{y}$

$$= \cancel{\frac{\partial B_z}{\partial y}}^{\,0}\hat{x} - \frac{\partial B_z}{\partial x}\hat{y}$$

no y variable in B_z

$$= \frac{\partial}{\partial x} B_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx)\,\hat{y}$$

let $B_0 = \frac{E_0}{c}$ and $k = \frac{\omega}{c}$

$$= -k B_0 \int_{-\infty}^{\infty} f(k)\sin(\omega t - kx)\,\hat{y}$$

$$= \frac{-E_0\omega}{c^2} \int_{-\infty}^{\infty} f(k)\sin(\omega t - kx)\,\hat{y}$$

let $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$$= -\mu_0\varepsilon_0 \cdot E_0\,\omega \int_{-\infty}^{\infty} f(k)\sin(\omega t - kx)\,\hat{y}$$

$$= (\mu_0\varepsilon_0)\left(-\frac{\partial}{\partial t} E_0 \int f(k)\cos(\omega t - kx)\,\hat{y}\right)$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \varepsilon_0 \cdot \frac{\partial \vec{E}}{\partial t}}$$

3) $\nabla \cdot \vec{E} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle E_x, E_y, E_z \right\rangle$

$$= \cancel{\frac{\partial E_x}{\partial x}}^{\,0} + \frac{\partial E_y}{\partial y} + \cancel{\frac{\partial E_z}{\partial z}}^{\,0}$$

(no component of $\hat{x}$ and $\hat{z}$ in $\vec{E}$)

$$= \frac{\partial}{\partial y}\left[E_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx) \right]$$

$= \emptyset$ (no y variable in E_y)

$$\boxed{\nabla \cdot \vec{E} = \emptyset}$$

4) $\nabla \cdot = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle B_x, B_y, B_z \right\rangle$

$$= \cancel{\frac{\partial B_x}{\partial x}}^{\,0} + \cancel{\frac{\partial B_y}{\partial y}}^{\,0} + \frac{\partial B_z}{\partial z}$$

(no component of $\hat{x}$ and $\hat{y}$ in $\vec{B}$)

$$= \frac{\partial}{\partial z}\left[B_0 \int_{-\infty}^{\infty} f(k)\cos(\omega t - kx) \right]$$

$= \emptyset$ (no z variable in B_z)

$$\boxed{\nabla \cdot \vec{B} = \emptyset}$$

3 Problem 3 **20 / 20**

- **- 5 pts** Did not verify Gauss' law for magnetic fields.
- **- 5 pts** Did not verify Faraday's law.
- **- 5 pts** Did not verify Ampere's law.
- **- 5 pts** Did not verify Gauss' law for electric fields.
- ✓ **- 0 pts** Correct
- **- 2.5 pts** Error verifying Ampere's law.
- **- 2.5 pts** Error verifying Gauss' law for electric fields.
- **- 2.5 pts** Error verifying Gauss' law for magnetic fields.
- **- 2.5 pts** Error verifying Faraday's law.