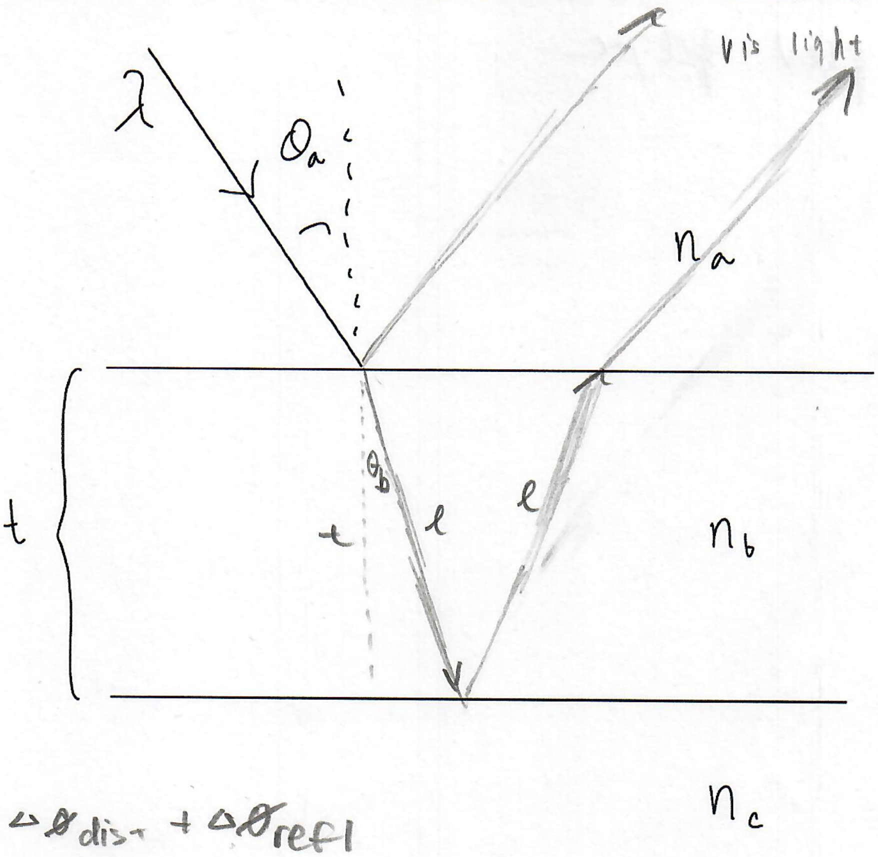
Problem 1

20 points

Consider a thin film of thickness $t = 2.30 \times 10^{-6}$ m and index of refraction $n_b = 1.20$. The film is resting on a material of index of refraction $n_c = 1.13$, and its top face is exposed to air $n_a = 1.00$. What is the shortest wavelength of *visible* light that will interfere destructively when incident on the film at angle $\theta_a = 22.0°$ from the normal? Give your answer in nanometers, to three significant figures. [Note: you may ignore the fact that the wavelength will change upon refraction; this will only very slightly affect the answer.]

vis light is 380-700nm



$n_b > n_a$ π shift for refl.

$n_c < n_b$ no π shift for refl.

$\Delta\theta_{tot} = \Delta\theta_{dist} + \Delta\theta_{refl}$ $\therefore \Delta\theta_{refl} = \pi$

$\frac{2\pi}{\lambda}\Delta x + \pi = (2m+1)\pi$

$\frac{4\pi l}{\lambda} + \pi = (2m+1)\pi$

$\frac{4\pi l}{\lambda} + \cancel{\pi} = 2m\pi + \cancel{\pi}$

$\frac{2l}{\lambda} = m$

$\frac{2t}{\lambda\cos\theta_b} = m$

$\lambda = \frac{2l}{m}$ as $m\uparrow$, $\lambda\downarrow$

try $\lambda = 380 \times 10^{-9}$ m

$m = \frac{2l}{\lambda} = 12.74$

13 would be too short of a wavelength so $m = 12$

find Δx

$n_a \sin\theta_a = n_b \sin\theta_b$

$\theta_b = \sin^{-1}\left(\frac{\sin\theta_a}{n_b}\right)$

$= \sin^{-1}\left(\frac{\sin(22)}{1.2}\right)$

$= 18.19$

$\Delta x = 2l$

$\cos\theta_b = \frac{t}{l}$

$l = \frac{t}{\cos\theta_b}$

$= \frac{2.3 \times 10^{-6}}{\cos(18.19)}$

$= 2.421 \times 10^{-6}$

$\boxed{404\text{nm}}$

$\lambda_{smallest} = \frac{2l}{12} = \frac{l}{6} = 4.03497 \times 10^{-7}$ m

~~404nm~~ $\boxed{403.5\text{nm}}$

This page intentionally left blank

see prev page

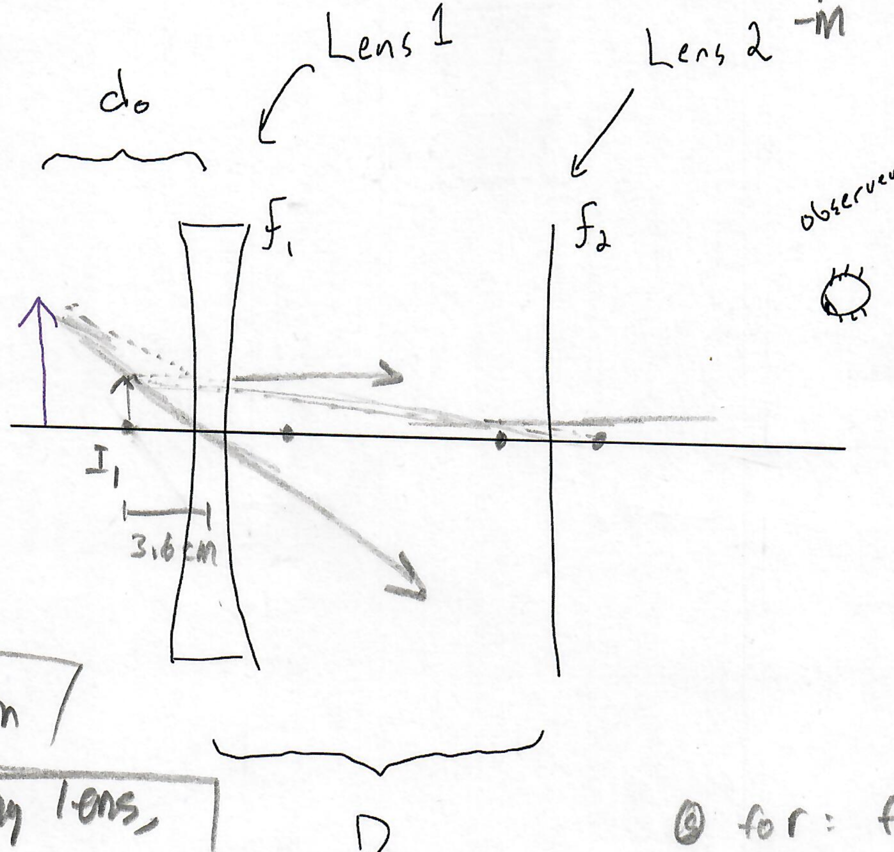
Problem 2

20 points

Consider the thin lens (Lens 1) of focal length $f_1 = -2.00$ cm shown below. An object is placed a distance $d_o = 4.50$ cm to the left of the lens. A second lens (Lens 2, drawn as a single vertical line in the diagram) is then placed $D = 5.00$ cm to the right of Lens 1. What is the focal length Lens 2 must have such that the *final* image of the object seen through the two lenses is inverted and enlarged by a factor of three with respect to the object? Give your answer in centimeters, to three significant figures. Is Lens 2 a converging or diverging lens?

converging

$-M$ $\quad M > 1 \quad M = 3$



$\boxed{f_2 = 6.79\text{cm}}$

converging lens, $f > 0$

$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$

$\frac{s-f}{sf} = \frac{1}{s'}$

$s' = \frac{sf}{s-f} = \frac{4.5(-2)}{4.5-2} = -3.6$

$s_1 = D - s' = 5 + 3.6$

$m = -\frac{s'}{s} = \frac{3.6}{4.5} = 0.8$

$m = \frac{y_1'}{y}$

$y_1' = 0.8y$

Ⓢ for: f_2 known: s_1, M_{10}, M_{20}

$\frac{y_2}{y_0} = -3$

$M_{12} = -\frac{s_2}{s_1}$

$\frac{1}{f_2} = \frac{1}{s_1} + \frac{1}{s_2}$

$f_2 = \frac{s_1 s_2}{s_1 + s_2}$

$s_2 = -M_{12} s_1 = 3.75(8.6) = 32.25$

$M_{12} = \frac{y_2}{y_1} = \frac{y_2}{0.8 y_0}$

$f_2 = \frac{32.25(8.6)}{8.6+32.25} = 6.789\text{cm}$

$0.8 M_{12} = \frac{y_2}{y_0} = -3$

$M_{12} = -3.75$

This page intentionally left blank

see prev page

Problem 3

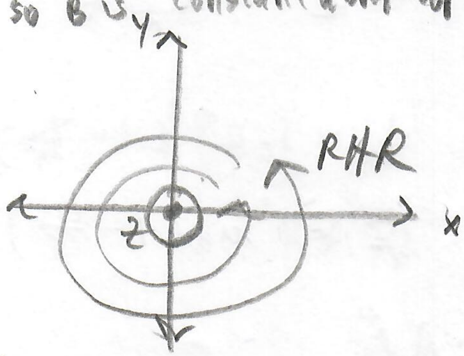
20 points

Consider an infinitely long cylindrical wire of radius R centered on the z axis. The current density in the wire varies with the distance ρ from the axis as $\vec{J}(\rho) = J_0\left(1 - \frac{\rho}{R}\right)\hat{z}$, where J_0 is a constant.

(a): 10 points

Find the magnetic field everywhere in space. Recall that current through a surface S is defined as $I = \int_S \vec{J} \cdot d\vec{A}$.

rotational sym
so B is constant along loop



$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

amp. curve of radius r

I_{encl} (total) $= \int_0^R J_0\left(1 - \frac{\rho}{R}\right) \cdot 2\pi\rho \, d\rho$

$= J_0 2\pi \int_0^R \rho - \frac{1}{R}\rho^2 \, d\rho$

$= 2\pi J_0 \left[\frac{1}{2}\rho^2 - \frac{1}{3R}\rho^3\right]_0^R$

$= 2\pi J_0 \left[\frac{1}{2}R^2 - \frac{1}{3R}R^3\right]$

$= 2\pi J_0 \left[R^2\left(\frac{1}{2} - \frac{1}{3}\right)\right]$

$= 2\pi J_0 R^2 \cdot \frac{1}{6}$

$= \frac{1}{3}\pi J_0 R^2$

$d\vec{A} = 2\pi\rho \, d\rho$

I_{encl} (inside wire) $= \int_0^\rho J_0\left(1 - \frac{\rho^*}{R}\right) \cdot 2\pi\rho^* \, d\rho^*$

$= 2\pi J_0 \left[\frac{1}{2}\rho^{*2} - \frac{1}{3R}\rho^{*3}\right]_0^\rho$

$= 2\pi J_0 \left[\frac{1}{2}\rho^2 - \frac{1}{3R}\rho^3\right]$

$\oint \vec{B} \cdot d\vec{l} = B \cdot 2\pi\rho$

inside

$B \cdot \cancel{2\pi}\rho = \cancel{2\pi} J_0 \left[\frac{1}{2}\rho^2 - \frac{1}{3R}\rho^3\right]\mu_0$

$B = \frac{J_0 \mu_0}{\rho}\left[\frac{1}{2}\rho^2 - \frac{1}{3R}\rho^3\right]$

$\boxed{B = \mu_0 J_0 \left[\frac{1}{2}\rho - \frac{1}{3R}\rho^2\right]}$

outside

$B \cdot 2\cancel{\pi}\rho = \frac{1}{3}\cancel{\pi} J_0 R^2 \mu_0$

$\boxed{B = \frac{1}{6\rho} J_0 R^2 \mu_0}$

direction: rotate cc around z-axis as shown in diagram

Problem 3 continued on next page...

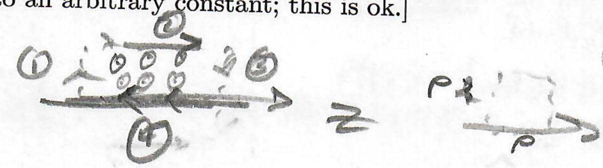
(b): 10 points

Suppose now that the current density falls off as a function of time:

$$\vec{J}(\rho, t) = J_0\left(1 - \frac{\rho}{R}\right)\left(1 - \frac{t}{\tau}\right)\hat{z}$$

where τ is a constant and $0 \le t \le \tau$. Calculate the electric field everywhere outside the wire, in terms of $\mu_0, \epsilon_0, J_0, R, \tau$, and any spatial coordinates and/or time. [Hint: the electric field will point along the z axis. Use Faraday's law and integrate around a rectangle whose base and top are parallel to the z axis.] [Note: you will only be able to calculate the electric field up to an arbitrary constant; this is ok.]

$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$



use square

1) find B

$I_{encl} = \int_0^R J_0(1-\frac{\rho}{R})(1-\frac{t}{\tau})\, 2\pi\rho\, d\rho$

$= (1-\frac{t}{\tau})\int_0^R J_0(1-\frac{\rho}{R})\, 2\pi\rho\, d\rho$

$= \frac{1}{3}\pi J_0 R^2 (1-\frac{t}{\tau})$

$B \cdot 2\pi\rho = \frac{1}{3}\pi J_0 R^2 (1-\frac{t}{\tau})\mu_0$

$B = \frac{1}{6\rho} J_0 R^2 \mu_0 (1-\frac{t}{\tau})$

2) find $\Phi_B(t)$

$d\Phi_B = \vec{B} \cdot d\vec{A}$

$\Phi_B = \int \vec{B} \cdot d\vec{A}$

$= \int \vec{B}\rho \cdot d\vec{\rho}$

$= \int \frac{1}{6} J_0 R^2 \mu_0 (1-\frac{t}{\tau})\, d\rho$

$= \frac{1}{6} J_0 R^2 \mu_0 \rho (1-\frac{t}{\tau})$

$\frac{d\Phi_B}{dt} = -\frac{1}{6\tau} J_0 R^2 \mu_0 \rho$

3) find $\oint \vec{E} \cdot d\vec{l}$

$\vec{E}$ is in $\hat{z}$ direc so

$\oint = 0$ along sides 1, 3

E is constant along the straight line. 0 along side 4 b/c no mag field there (no charge encl @ middle of wire)

$\therefore \oint \vec{E} \cdot d\vec{\rho} = E\rho$

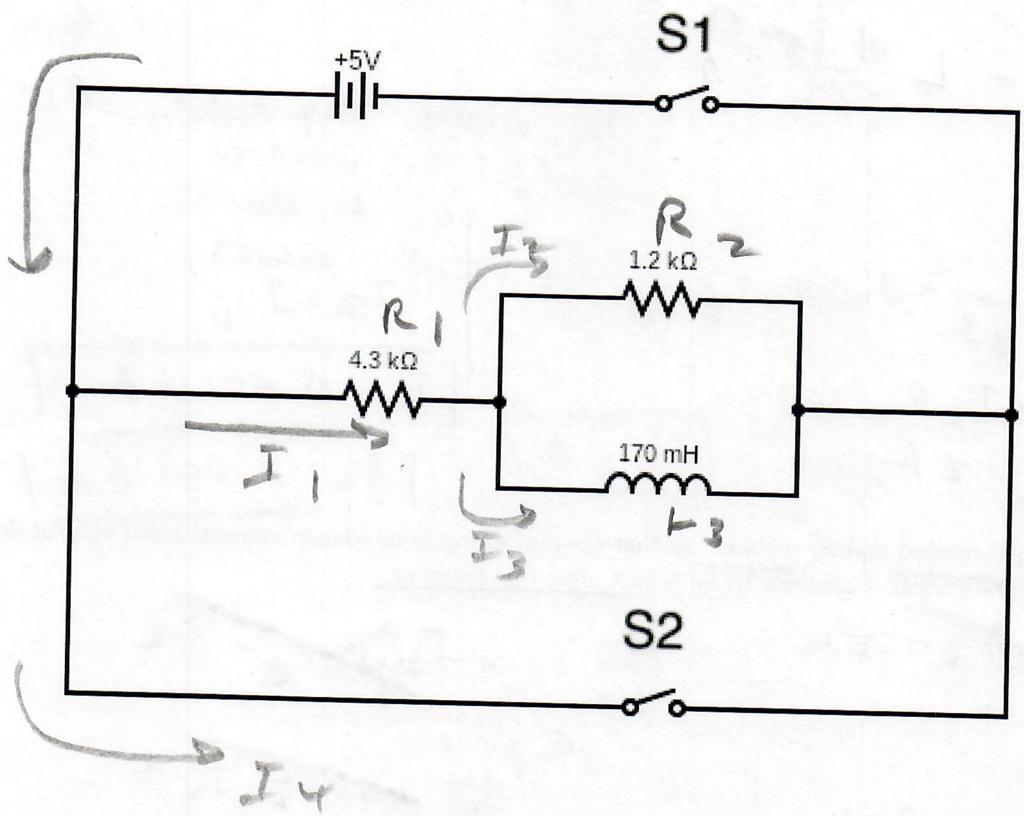
$E\cancel{\rho} = \frac{1}{6\tau} J_0 R^2 \mu_0 \cancel{\rho}$

$$\boxed{E = \frac{1}{6\tau} J_0 \mu_0 R^2}$$

Problem 4

20 points

Consider the circuit drawn below. Let I_1 be the current flowing through the 4.3 kΩ resistor,, I_2 be the current flowing through the 1.2 kΩ resistor, and I_3 be the current flowing through the 170 mH inductor.



Problem 4 continued on next page. . .

(a): 5 points

Suppose S_1 has been closed for a long time, $(-\infty < t < 0)$, and the emf source drives a steady current. Calculate I_1, I_2, I_3.

$\frac{di}{dt} = 0$

$\mathcal{E} - I_1 R_1 - I_2 R_2 = 0$

$I_1 = I_2 + I_3$

$\mathcal{E} - I_1 R_1 - L\frac{dI_3}{dt} = 0$

$\mathcal{E} = 5$

$R_1 = 4.3\ k\Omega$

no voltage drop across inductor when ~~the~~ $\frac{di}{dt} = 0$ so short circuit occurs

$\mathcal{E} = I_1 R_1$

$I_1 = \frac{.5}{4.3 \times 10^3} = 0.001163\,A$

$I_3 = I_1$

$\boxed{I_1 = 0.00116\,A}$

$\mathcal{E} - \mathcal{E} - I_2 R_2 = 0$

$I_2 R_2 = 0$

$\boxed{I_2 = 0}$

$\boxed{I_3 = 0.00116\,A}$

(b): 5 points

At time $t = 0$, S_1 is opened and S_2 is closed, so that the emf source is no longer connected and current flows through the bottom branch. Calculate I_1, I_2, I_3 just after this happens.

$I_3 = \cancel{I_3^* e^{-\frac{R}{L}t}}$

from part a

$\frac{dI_3}{dt} = \cancel{-\frac{I_3^* R}{L} e^{-\frac{R}{L}t}}$

@ $t = 0$

$\frac{dI_3}{dt} = \cancel{-\frac{R}{L} I_3^*}$

$I_3 = I_3^* e^0 = I_3^*$

$L\frac{dI_3}{dt} = I_1 R_1$

$I_1 = I_2 + I_3$

$\int \frac{1}{I_1} dI_3 \int \frac{R_1}{L}\, dt$???

$L\frac{dI_3}{dt} - I_1 R_1 = 0$

$I_1 R_1 = L\frac{dI_3}{dt}$

$I_1(I_3) = I_2 + I_3$

current is dec so this term is +

$\cancel{I_1 = -\frac{L}{R_1} \cdot \frac{R}{L} I_3^*}$

@ $t=0$, no time to discharge so current does not charge?

$\boxed{I_1 = 0.00116\,A}$ $\boxed{I_3 = 0.00116\,A}$ $\boxed{I_2 = 0}$

(c): 5 points

Calculate I_1, I_2, I_3 in the limit $t \to \infty$.

as $t \to \infty$, $\frac{dI_3}{dt} \to 0$

system is discharged so $I_3 = 0$

$L \cancel{\frac{dI_3}{dt}}^{\,0} - I_1 R_1 = 0 \qquad I_1 = 0$

$I_1 = I_2 + I_3$

$$\boxed{I_1 = 0A,\ I_2 = 0\ A,\ I_3 = 0\ A}$$

(d): 5 points

Calculate the total energy dissipated in all resistors during the time period $0 < t < \infty$.

$U = \frac{1}{2} L i^2$ → must be equal to all energy present @ the start

energy stored in inductor

$= \frac{1}{2}(170 \times 10^{-3})(0.00116)^2$

$= \boxed{1.14376 \times 10^{-7}\ J}$

Problem 5

10 points

(a): 5 points

Consider the following vector field:

$$\vec{V}(r, \theta, \phi) = \alpha \frac{1}{r^2} \hat{r},$$

where α is a constant with units T·m^2, r is the distance from the origin, and $\hat{r}$ is a unit vector pointing away from the origin. According to Maxwell's equations, could this represent a magnetic field? Explain.

$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\quad \nabla \cdot \vec{E} = 0$ $\quad \nabla \cdot \vec{B} = 0$

$$\nabla \vec{V} = \left\langle \frac{\partial \vec{V}}{\partial r}, \frac{\partial \vec{V}}{\partial \theta}, \frac{\partial \vec{V}}{\partial \theta} \right\rangle$$

$$= \left\langle -\frac{2\alpha}{r^3}, 0, 0 \right\rangle$$

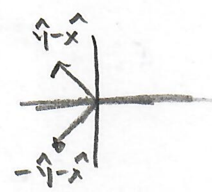
$$\nabla \vec{V} \cdot \vec{V} = -\frac{2\alpha}{r^3}\left(\alpha \frac{1}{r^2}\right) = -\frac{2\alpha^2}{r^5} \neq 0$$

this vector field does not meet the divergence condition and $\therefore$ cannot be a magnetic field

(b): 5 points

Consider the following electric and magnetic fields:

$$\vec{E} = \frac{E_0}{\sqrt{2}} \cos\left(\omega(t - z/c)\right)(\hat{y} - \hat{x})$$
$$\vec{B} = \frac{E_0}{c\sqrt{2}} \cos\left(\omega(t - z/c)\right)(-\hat{y} - \hat{x}).$$



Can these fields constitute a traveling electromagnetic plane wave in vacuum? If yes, prove it. If not, explain why not.

- $B_0 = \frac{E_0}{c}$ - amplitudes are proportional
- waves travel in the same direction (z) and are in phase (same cos arg.)

- $\hat{y} - \hat{x}$ is $\perp$ to $-\hat{y} - \hat{x}$ and each is $\perp$ to the z prop. direction

the two fields meet all necessary conditions so they can constitute a traveling EM plane wave

Problem 6

10 points

Consider an LRC series circuit driven with an ac source $v(t) = V_0 \cos(\omega t)$. You may use without proof the impedance and phase of an LRC series circuit.

(a): 5 points

Calculate the total energy dissipated in the resistor over one cycle if the system is driven at its resonant frequency. $X_L = X_C$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$V_0 = I\sqrt{R^2 + (X_L - X_C)^2} = IZ = IR$$

$$P_R = i^2 R$$

$$i(t) = I\cos(\omega_0 t)$$

$$P_{av} = \frac{1}{2} I^2 R$$

$$P_R = I^2 R \cos^2(\omega_0 t) \qquad T = \frac{2\pi}{\omega}$$

$$\tan\theta = \frac{X_L - X_C}{R}$$

$$\theta = 0, \quad X_L - X_C = 0$$

$$\therefore \quad i(t) = I\cos(\omega_0 t)$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= \cos^2 x - (1 - \cos^2 x)$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\frac{1}{2}(\cos 2x + 1) = \cos^2 x$$

$$P_{tot} = \int_0^{\frac{2\pi}{\omega}} I^2 R \cos^2(\omega_0 t)\, dt$$

$$= I^2 R \int_0^{\frac{2\pi}{\omega_0}} \cos^2(\omega_0 t)\, dt$$

$$= I^2 R \int_0^{\frac{2\pi}{\omega_0}} \frac{1}{2}(\cos(2\omega_0 t) + 1)\, dt$$

$$= \frac{1}{2} I^2 R \int_0^{\frac{2\pi}{\omega_0}} \cos(2\omega_0 t) + 1\, dt$$

$$= \frac{1}{2} I^2 R \left[\frac{1}{2\omega_0}\sin(2\omega_0 t) + t\right]_0^{2\pi/\omega_0}$$

$$= \frac{1}{2} I^2 R \left[\cancel{\frac{1}{2\omega_0}\sin(4\pi)}^{\,0} + \frac{2\pi}{\omega_0} - 0\right]$$

$$= \frac{\pi}{\omega_0} I^2 R \qquad [\text{just } P_{av} \cdot T]$$

$$= \frac{R\pi}{\omega_0}\left(\frac{V_0}{R}\right)^2$$

$$= \frac{V_0^2 \pi}{R\omega_0}$$

$$\boxed{P_{tot} = \frac{V_0^2 \pi}{R\omega_0}}$$

Problem 6 continued on next page...

(b): 5 points

Calculate the total energy dissipated in the resistor over one cycle if the system is driven at *twice* its resonant frequency. Is it greater than or less than the energy found in part (a)?

$$Z_{LRC} = \sqrt{R^2 + (\omega L - (1/\omega C))^2} \qquad \omega = 2\omega_0$$

$$2\omega_0 L - \frac{1}{2\omega_0 C} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$= \frac{2L}{\sqrt{LC}} - \frac{\sqrt{LC}}{2C} \qquad I Z_{LRC} = V_0$$

$$= \frac{4LC}{2C\sqrt{LC}} - \frac{LC}{2C\sqrt{LC}} \qquad I = \frac{V_0}{Z_{LRC}}$$

$$= \frac{3LC}{2C\sqrt{LC}} \qquad P_{av} = \frac{1}{2} I^2 R$$

Z_{LRC}

$$= \frac{3L}{2\sqrt{LC}} \qquad = \frac{1}{2}\left(\frac{V_0}{\sqrt{R^2 + \frac{9L}{4C}}}\right)^2 R$$

$$= \frac{1}{2} R \cdot \frac{V_0^2}{R^2 + \frac{9L}{4C}}$$

$$Z_{LRC} = \sqrt{R^2 + \left(\frac{3L}{2\sqrt{LC}}\right)^2}$$

$$= \sqrt{R^2 + \frac{9L^2}{4LC}} \qquad P_{tot} = P_{av} \cdot \frac{2\pi}{2\omega_0}$$

$$= \sqrt{R^2 + \frac{9L}{4C}} \qquad = P_{av} \cdot \frac{\pi}{\omega_0}$$

$$= \frac{RV_0^2}{2(R^2 + \frac{9L}{4C})} \cdot \frac{\pi}{\omega_0}$$

$$= \boxed{\frac{V_0^2 \pi}{\omega_0} \cdot \frac{R}{2(R^2 + \frac{9L}{4C})}}$$

since $\frac{1}{R}$ is greater than $\frac{R}{2(R^2 + \frac{9L}{4C})}$ more energy is dissip in resistor when system is @ resonant freq

conclusion makes sense b/c ω_0 minimizes impedence contrib. from L and C

compare to

$$\frac{V_0^2 \pi}{\omega_0} \cdot \frac{1}{R} \qquad \frac{1}{R} > \frac{R}{2(R^2 + \frac{9L}{4C})}$$

$$2R^2 + \frac{9L}{2C} > R^2$$

b/c $\frac{9L}{2C}$ is always positive. also $\times 2$ R^2

$$\frac{1}{R^2} > \frac{1}{2(R^2 + \frac{9L}{4C})}$$