Physics 1C - Spring 2022: Midterm 2 Solutions

Problem 1

(34/100)

A parabolic antenna with a diameter of 20.0 m receives (at normal incidence) a radio signal from a distant transmission tower. The radio signal is a continuous sinusoidal wave with amplitude $E_{\max} = 2.00\times10^{-7}$ V/m. Assume that the antenna completely absorbs all of the radiation incident upon it. (Note: $c = 3.00\times10^{8}$ m/s, $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2}$, $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T\cdot m/A}$)

(a) (6 points) What is the amplitude of the magnetic field in this wave?

(b) (8 points) Find the intensity of the radiation received by this antenna.

(c) (8 points) Determine the power received by the antenna.

(d) (8 points) What is the force exerted by the radio waves on the antenna?

(e) (4 points) Suppose the radio waves are traveling in the x-direction and are polarized so that the electric field points in the direction $(\hat{\mathbf{j}}+\hat{\mathbf{k}})/\sqrt{2}$. What is the direction of the magnetic field for the radio waves?

Solution:

(a) The amplitude of the magnetic field is

$$B_{\max} = \frac{E_{\max}}{c} = \frac{2.00 \times 10^{-7}\ \mathrm{V/m}}{3.00 \times 10^{8}\ \mathrm{m/s}} = 6.67 \times 10^{-16}\ \mathrm{T}.$$

(b) For the intensity, we have

$$I = S_{\mathrm{avg}} = \frac{E_{\max}^2}{2\mu_0 c} = \frac{(2.00 \times 10^{-7}\ \mathrm{V/m})^2}{2(4\pi \times 10^{-7}\ \mathrm{T\cdot m/A})(3.00 \times 10^{8}\ \mathrm{m/s})} = 5.31 \times 10^{-17}\ \mathrm{W/m^2}.$$

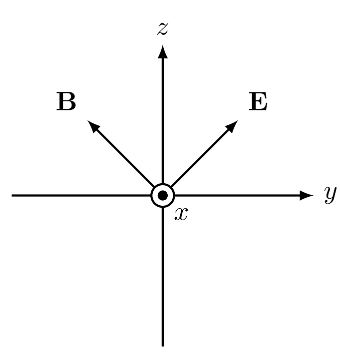
(c) Since the antenna has a cross sectional area of $A = \pi r^2$, where $r = d/2 = 10.0$ m, the average power received by the antenna is

$$P = IA = I\pi r^2 = (5.31 \times 10^{-17}\ \mathrm{W/m^2})\pi(10.0\ \mathrm{m})^2 = 1.67 \times 10^{-14}\ \mathrm{W}.$$

(d) The antenna completely absorbs all of the radiation from the radio signals, so the radiation pressure is $p_{\mathrm{rad}} = I/c$. The total force F exerted on the antenna is therefore

$$F = p_{\mathrm{rad}}A = \frac{IA}{c} = \frac{1.67 \times 10^{-14}\ \mathrm{W}}{3.00 \times 10^{8}\ \mathrm{m/s}} = 5.56 \times 10^{-23}\ \mathrm{N}.$$

(e) Recall that electromagnetic waves propagate in the direction of $\mathbf{E} \times \mathbf{B}$, and in this case the wave is propagating in the $\hat{\mathbf{i}}$ direction. The simplest way to determine the direction of $\mathbf{B}$ is to use the right hand rule and point your index finger in the direction of $\mathbf{E}$ and your thumb in the direction of propagation for the wave. This tells us that the direction of $\mathbf{B}$ must be $(-\hat{\mathbf{j}} + \hat{\mathbf{k}})/\sqrt{2}$:



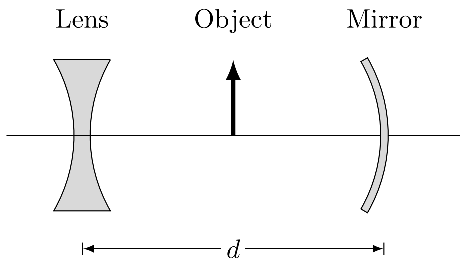
To verify this, you can check that

$$\frac{1}{2}(\hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (-\hat{\mathbf{j}} + \hat{\mathbf{k}}) = \frac{1}{2}(\hat{\mathbf{j}} \times \hat{\mathbf{k}}) - \frac{1}{2}(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) = \frac{1}{2}\,\hat{\mathbf{i}} + \frac{1}{2}\,\hat{\mathbf{i}} = \hat{\mathbf{i}}.$$

Problem 2

(36/100)

An object is placed midway between a diverging lens and a mirror, which are separated by a distance $d = 25.0$ cm. The magnitude of the mirror's radius of curvature is 20.0 cm, and the lens has a focal length of -16.7 cm.



(a) (20 points) Considering only the light that leaves the object and travels first toward the mirror, locate the final image formed by this system after the light reflects off of the mirror and passes through the lens.

(b) (4 points) Is the image real or virtual?

(c) (8 points) What is the overall magnification?

(d) (4 points) Is the image upright or inverted?

Solution:

(a) We shall denote the object distance with respect to the mirror as s_1, and since the object is halfway between the mirror and the lens, $s_1 = 12.5$ cm. Because the mirror's radius of curvature is $R = 20.0$ cm, the focal length for the mirror is $f_1 = R/2 = 10.0$ cm. The light is first directed towards the mirror and reflects off of it, which gives us an image distance of

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \quad \rightarrow \quad s_1' = \left(\frac{1}{f_1} - \frac{1}{s_1}\right)^{-1} = \left(\frac{1}{10.0 \text{ cm}} - \frac{1}{12.5 \text{ cm}}\right)^{-1} = 50.0 \text{ cm}.$$

This forms a real image that is to the left of the mirror, which serves as the object for the diverging lens next to the mirror.
But since the image s_1' is 50.0 cm to the left of the mirror, it is a distance $s_2 = 25.0 \text{ cm} - 50.0 \text{ cm} = -25.0$ cm to the left of the lens, which means it is a virtual object. The image formed by the lens s_2' is then

$$s_2' = \left(\frac{1}{f_2} - \frac{1}{s_2}\right)^{-1} = \left(\frac{1}{-16.7 \text{ cm}} - \frac{1}{-25.0 \text{ cm}}\right)^{-1} = -50.3 \text{ cm}.$$

Thus, the final image is formed $-50.3 \text{ cm} + 25 \text{ cm} = -25.3$ cm to the right of the mirror.

(b) The final image distance is negative, which means it is a virtual image.

(c) To find the overall magnification, we must first find the individual magnification factors m_1 and m_2. We have

$$m_1 = -\frac{s_1'}{s_1} = -\frac{50.0 \text{ cm}}{12.5 \text{ cm}} = -4.00, \quad m_2 = -\frac{s_2'}{s_2} = -\frac{(-50.3 \text{ cm})}{(-25.0 \text{ cm})} = -2.01.$$

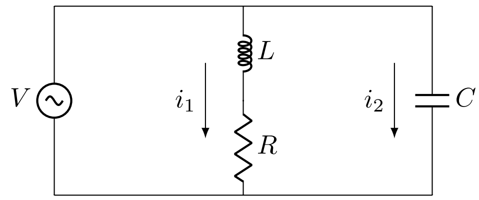
So the overall magnification is

$$m = m_1 m_2 = (-4.00)(-2.01) = 8.05.$$

(d) The magnification m is positive, so the image is upright.

Problem 3 (30/100)

An inductor L and a resistor R are connected in series to an ac voltage source with amplitude V. This circuit is then connected in parallel to a capacitor with capacitance $C = L/2R^2$. The angular frequency ω of the voltage source is chosen so that $\omega = R/\sqrt{3}L$, and the current i_1 passing through the inductor and resistor is given by $i_1 = I_1 \cos \omega t$, while the current passing through the capacitor is denoted by i_2.



(a) (6 points) Find the amplitudes I_1 and I_2 of the two currents in the circuit in terms of V and R.

(b) (10 points) Obtain an expression for the total instantaneous current $i = i_1 + i_2$ passing through the voltage source. (Hint: You might want to write the expression for i in terms of a sum of cosines. A useful trig identity is $-\sin x = \cos(x + \pi/2)$.)

(c) (6 points) Draw a phasor diagram for the voltages and currents.

(d) (8 points) Determine the current amplitude I for the current passing through the voltage source in terms of V and R. (Hint: Use the fact that $I^2 = I_1^2 + I_2^2 + 2I_1 I_2 \cos \alpha$, where α is the angle between the phasors for I_1 and I_2.)

Solution:

(a) For current i_1, we only need to consider the inductor, resistor, and voltage source. This is essentially an ac RL circuit, which means that the impedance Z_1 of this branch of the circuit is

$$Z_1 = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2} = \sqrt{R^2 + \frac{R^2}{3}} = \frac{2R}{\sqrt{3}}.$$

Therefore, the current amplitude for i_1 is

$$I_1 = \frac{V}{Z_1} = \frac{\sqrt{3}V}{2R}.$$

For the current through the capacitor, the voltage amplitude is V because it is connected in parallel with the voltage source. This can also be treated as a separate ac circuit, with impedance

$$Z_2 = X_C = \frac{1}{\omega C} = \frac{\sqrt{3}L}{R}\frac{2R^2}{L} = 2\sqrt{3}R.$$

The amplitude of the current i_2 is therefore

$$I_2 = \frac{V}{Z_2} = \frac{V}{2\sqrt{3}R}.$$

(b) We are given that $i_1 = I_1 \cos \omega t$, so we know that

$$i_1 = \frac{\sqrt{3}V}{2R} \cos \omega t.$$

Since the left half of the circuit is an ac RL circuit, we can write down the voltage from the ac source as $v = V \cos(\omega t + \phi)$. But this is also the voltage across the capacitor since it is connected in parallel with the voltage source, so we know that

$$v_C = \frac{q}{R} = V \cos(\omega t + \phi).$$

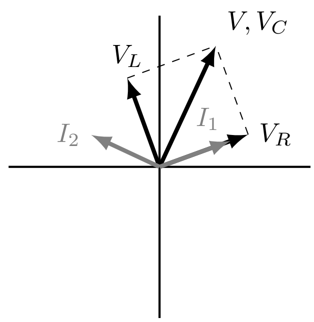
The current through the capacitor is therefore

$$i_2 = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[CV\cos(\omega t + \phi)] = -\omega CV \sin(\omega t + \phi) = \frac{V}{2\sqrt{3}R}\cos(\omega t + \phi + \pi/2),$$

where we have used the fact that $-\sin x = \cos(x + \pi/2)$. The total current is then

$$i = \frac{\sqrt{3}V}{2R}\cos\omega t + \frac{V}{2\sqrt{3}R}\cos(\omega t + \phi + \pi/2) = \frac{V}{R}\left[\frac{\sqrt{3}}{2}\cos\omega t + \frac{1}{2\sqrt{3}}\cos(\omega t + \phi + \pi/2)\right].$$

(c) First let's consider the voltage phasors. There is an inductor and a resistor, with the inductor having voltage amplitude $V_L = I\omega L$ and the resistor having voltage amplitude $V_R = I_R R$. The voltage phasor for the inductor leads the resistor's phasor by 90°, and the vector sum of these two phasors must equal the magnitude of the voltage amplitude V. Furthermore, since the capacitor is in parallel with the left half of the circuit, its phasor is in phase with the voltage phasor, and its amplitude $V_C = I_C X_C$ is the same as V.
For the currents, the phasor for i_1 has amplitude I_1 and is in phase with the voltage phasor for the resistor. Meanwhile, the current phasor for i_2 has amplitude I_2 and is ahead of the voltage phasor for the capacitor by 90°. Therefore, the phasor diagram looks as follows:



(d) The current amplitude for the current passing through the voltage source is simply the amplitude of the total current $i = i_1 + i_2$. To find I, we must take the vector sum of the two current phasors for I_1 and I_2. The law of cosines tells us that

$$I^2 = I_1^2 + I_2^2 + 2I_1I_2\cos\alpha,$$

where α is the phase angle between the two current phasors.
The current i_1 is proportional to $\cos\omega t$, while i_2 is proportional to $\cos(\omega t + \phi + \pi/2)$, so the phase angle between the phasors is $\alpha = \phi + \pi/2$. Therefore,

$$I^2 = I_1^2 + I_2^2 + 2I_1I_2\cos(\phi + \pi/2) = I_1^2 + I_2^2 - 2I_1I_2\sin\phi,$$

where we have used the fact that $-\sin x = \cos(x + \pi/2)$. It then remains to determine $\sin\phi$. The angle ϕ is the phase angle for the left branch of the circuit, and since that branch is an ac RL circuit, we know that the phase angle is

$$\tan\phi = \frac{X_L}{R} = \frac{\omega L}{R} = \frac{1}{\sqrt{3}},$$

which means $\phi = 30°$. Therefore, $\sin\phi = \sin(30°) = 1/2$, which gives us

$$I^2 = I_1^2 + I_2^2 - I_1I_2 = \frac{3V^2}{4R^2} + \frac{V^2}{12R^2} - \frac{V^2}{4R^2} = \frac{V^2}{R^2}\left(\frac{3}{4} + \frac{1}{12} - \frac{1}{4}\right) = \frac{7V^2}{12R^2},$$

so the current amplitude is

$$I = \sqrt{\frac{7}{12}}\frac{V}{R}.$$