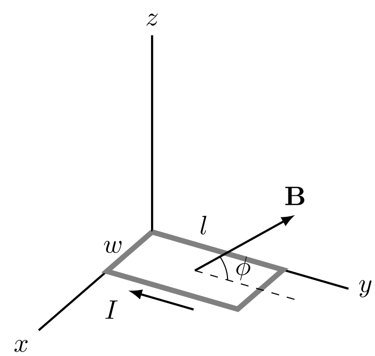
Physics 1C - Spring 2022: Midterm 1 Solutions

Problem 1

(34/100)

A rectangular loop of wire is $l = 0.500$ m long by $w = 0.300$ m wide and lies in the xy-plane, as shown in the figure below. A uniform magnetic field $\mathbf{B}$ with magnitude 1.50 T is directed into the loop at an angle of $\phi = 40.0°$ with respect to the plane of the loop, with the magnetic field lines parallel to the yz-plane. The loop carries a current of $I = 0.900$ A in the direction shown.



(a) (12 points) What are the magnitudes of the forces on the four wire segments of lengths w and l?

(b) (9 points) What is the magnitude and direction of magnetic moment $\boldsymbol{\mu}$ of the wire loop?

(c) (9 points) Find the magnitude of the torque τ on the wire loop due to the magnetic field.

(d) (4 points) About which axis will the wire loop rotate due to the torque?

Solution:

(a) For the two wire segments of length w, the current is perpendicular to the magnetic field, and so the force F_w for each of these segments is

$$F_w = IwB = (0.900 \text{ A})(0.300 \text{ m})(1.50 \text{ T}) = 0.405 \text{ N},$$

with the forces being equal and opposite.
For the segments of length l, there is a component of $\mathbf{B}$ that is either parallel or antiparallel to the current I. The magnitude of each force is

$$F_l = IlB \sin\phi = (0.900 \text{ A})(0.500 \text{ m})(1.50 \text{ T}) \sin(40.0°) = 0.434 \text{ N},$$

again with both forces being equal and opposite.

(b) The magnetic moment is defined as $\boldsymbol{\mu} = I\mathbf{A}$, where $\mathbf{A}$ is the area vector with magnitude $A = lw$ that points perpendicular to the plane of the current loop. The right hand rule tells us that since the current runs in the clockwise direction in the xy-plane, the area vector $\mathbf{A}$ must point in the negative z-direction, and so we have

$$\boldsymbol{\mu} = Ilw(-\hat{\mathbf{k}}) = -(0.900 \text{ A})(0.500 \text{ m})(0.300 \text{ m})\,\hat{\mathbf{k}} = -(0.135 \text{ A}\cdot\text{m}^2)\,\hat{\mathbf{k}}.$$

(c) The torque is defined by $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$, which has magnitude $\tau = \mu B|\sin\theta|$, where θ is the angle between $\boldsymbol{\mu}$ and $\mathbf{B}$. Since $\boldsymbol{\mu}$ points in the $-\hat{\mathbf{k}}$ direction, the angle θ is $\theta = \phi + 90°$. Therefore,

$$\tau = \mu B|\sin\theta| = (0.135 \text{ A}\cdot\text{m}^2)(1.50 \text{ T})|\sin(130°)| = 0.155 \text{ N}\cdot\text{m}.$$

(d) The torque is such that it causes the magnetic moment $\boldsymbol{\mu}$ to align with $\mathbf{B}$, which means that the loop will rotate about the x-axis. Alternatively, we could also compute the torque explicitly by using the fact that $\mathbf{B} = B(\cos\phi\,\hat{\mathbf{j}} + \sin\phi\,\hat{\mathbf{k}})$, from which we obtain

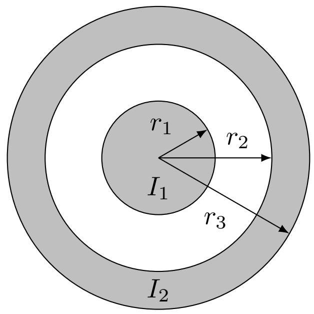
$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} = \mu B\Big[\cos\phi(-\,\hat{\mathbf{k}} \times \hat{\mathbf{j}}) + \sin\phi(-\,\hat{\mathbf{k}} \times \hat{\mathbf{k}})\Big] = \mu B\cos\phi\,\hat{\mathbf{i}},$$

which is in the positive x-direction.

Problem 2

(36/100)

Shown below is a cross sectional view of a coaxial cable consisting of a cylindrical core of radius $r_1 = 0.75$ mm and an outer shell with inner radius $r_2 = 1.50$ mm and outer radius $r_3 = 2.00$ mm. The inner conductor carries a current of $I_1 = 1.00$ A out of the page, while the current in the outer conductor carries a current of $I_2 = 3.00$ A into the page. (Note: $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$)



(a) Find the magnitude and direction (clockwise or counterclockwise) of the magnetic field at the following distances:

- (8 points) A distance of $r = 0.50$ mm away from the center.
- (8 points) A distance of $r = 1.25$ mm away from the center.
- (8 points) A distance of $r = 1.75$ mm away from the center.
- (8 points) A distance of $r = 2.50$ mm away from the center.

(b) (4 points) Suppose we can tune the value of the current I_2 in the outer shell. What must the value of I_2 be to ensure that there is no magnetic field when $r > r_3$?

Solution:

(a) For each of the distances, we apply Ampère's law to determine the magnitude of the magnetic field. As for the direction of the magnetic field, we shall take the counterclockwise direction to be positive, and the clockwise direction to be negative, so that the magnetic field contributions due to current I_1 are positive, and the contributions due to I_2 are negative.

- For the inner conductor of radius r_1, the current density is

$$J_1 = \frac{I_1}{\pi r_1^2},$$

and for $r = 0.50$ mm, $r < r_1$. Drawing an Ampèrian loop of radius r centered about the cable, the current enclosed is

$$I_{\text{enc}} = \int \mathbf{J}_1 \cdot \mathrm{d}\mathbf{A} = J_1(\pi r^2) = \frac{r^2}{r_1^2} I_1.$$

Meanwhile, the line integral of $\mathbf{B}$ yields

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B \oint \mathrm{d}l = B(2\pi r).$$

Therefore, we obtain

$$\begin{aligned} B(2\pi r) &= \frac{r^2}{r_1^2}\mu_0 I_1 \\ B &= \frac{\mu_0 I_1 r}{2\pi r_1^2} \\ &= \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(1.00\ \mathrm{A})(5.00 \times 10^{-4}\ \mathrm{m})}{2\pi(7.50 \times 10^{-4}\ \mathrm{m})^2} \\ &= 1.78 \times 10^{-4}\ \mathrm{T}, \end{aligned}$$

which is positive, and therefore points in the counterclockwise direction.

- At $r = 1.25$ mm, we are in the region $r_1 < r < r_2$, so the enclosed current is simply I_1. Therefore,

$$B(2\pi r) = \mu_0 I_1 \quad \rightarrow \quad B = \frac{\mu_0 I_1}{2\pi r} = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(1.00\ \mathrm{A})}{2\pi(1.25 \times 10^{-3}\ \mathrm{m})} = 1.60 \times 10^{-4}\ \mathrm{T},$$

  which also points in the counterclockwise direction.

- For $r = 1.75$ mm, we are now in the region $r_2 < r < r_3$, so our Ampèrian loop will enclose the current I_1, and some of I_2. Since I_2 is pointing into the page, we shall take its current contribution to be negative, which also takes into account the direction of its contribution to the magnetic field.

  First, we must determine the current density J_2 for the outer shell. The area of the shell is $\pi(r_3^2 - r_2^2)$, so the current density is

$$J_2 = \frac{I_2}{\pi(r_3^2 - r_2^2)}.$$

  Our Ampèrian loop of radius r encloses an area of $\pi(r^2 - r_2^2)$ for the outer shell, so the current enclosed by our Ampèrian loop is

$$I_{\mathrm{enc}} = I_1 - J_2(\pi r^2 - r_2^2) = I_1 - \frac{r^2 - r_2^2}{r_3^2 - r_2^2} I_2.$$

  The line integral of $\mathbf{B}$ is the same as before, so we have

$$\begin{aligned} B(2\pi r) &= \mu_0 \left(I_1 - \frac{r^2 - r_2^2}{r_3^2 - r_2^2} I_2 \right) \\ B &= \frac{\mu_0}{2\pi r} \left(I_1 - \frac{r^2 - r_2^2}{r_3^2 - r_2^2} I_2 \right) \\ &= \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})}{2\pi(1.75 \times 10^{-3}\ \mathrm{m})} \left[(1.00\ \mathrm{A}) - \frac{(1.75 \times 10^{-3}\ \mathrm{m})^2 - (1.50 \times 10^{-3}\ \mathrm{m})^2}{(2.00 \times 10^{-3}\ \mathrm{m})^2 - (1.50 \times 10^{-3}\ \mathrm{m})^2} (3.00\ \mathrm{A}) \right] \\ &= -4.49 \times 10^{-5}\ \mathrm{T}, \end{aligned}$$

  which is negative and therefore points in the clockwise direction.

- At $r = 2.50$ mm, our Ampèrian loop encloses the entirety of the cable, so the enclosed current is

$$I_{\mathrm{enc}} = I_1 - I_2,$$

  which corresponds to a magnetic field of

$$\begin{aligned} B(2\pi r) &= \mu_0 (I_1 - I_2) \\ B &= \frac{\mu_0 (I_1 - I_2)}{2\pi r} \\ &= \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})[(1.00\ \mathrm{A}) - (3.00\ \mathrm{A})]}{2\pi(2.50 \times 10^{-3}\ \mathrm{m})} \\ &= -1.60 \times 10^{-4}\ \mathrm{T}, \end{aligned}$$

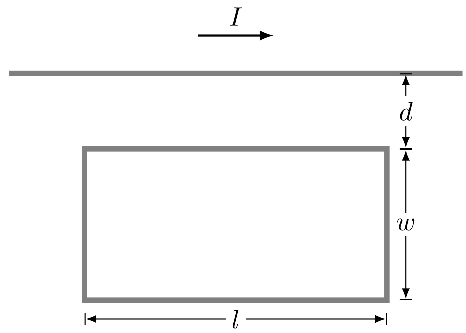
  which is in the clockwise direction.

(b) To ensure that there is no magnetic field when $r > r_3$, we need I_2 to cancel out I_1 exactly. Therefore, we require that $I_2 = I_1 = 1.00$ A.

Problem 3

(30/100)

A rectangular loop of wire with width w and length l and a long, straight wire carrying a current I are positioned next to each other and separated by a distance d, as indicated in the figure below. (Note: $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$)



(a) (14 points) Determine the magnetic flux through the loop due to the magnetic field generated by the wire with current I.

(b) (12 points) Suppose the current is changing with time, with the current given by

$$I(t) = a + bt,$$

where a and b are constants. Determine the emf that is induced in the loop if $a = 5.0$ A, $b = 10.0$ A/s, $d = 1.00$ cm, $w = 10.0$ cm, and $l = 1.00$ m.

(c) (4 points) What is the direction of the induced current in the rectangle?

Solution:

(a) The magnetic field due to the wire is given by

$$B = \frac{\mu_0 I}{2\pi y},$$

and since B points into the page in the region bounded by the rectangular loop, we shall take the area element $\mathrm{d}\mathbf{A} = -\,\mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{k}}$ of the loop to point into the page as well. Since the magnetic field passing through the loop is not uniform, we must integrate over the area of the rectangle, with the limits of integration going from 0 to l in the x-direction, and from d to $d + w$ in the y-direction. Then the flux through the loop due to the magnetic field is

$$\Phi_B = \int \mathbf{B} \cdot \mathrm{d}\mathbf{A} = \int_d^{w+d} \int_0^l \frac{\mu_0 I}{2\pi y}\,\mathrm{d}x\,\mathrm{d}y = \frac{\mu_0 I l}{2\pi} \int_d^{w+d} \frac{\mathrm{d}y}{y} = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{w+d}{d}\right).$$

(b) The emf through the rectangular loop is

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mu_0 I l}{2\pi} \ln\left(\frac{w+d}{d}\right)\right] = -\frac{\mu_0 l}{2\pi} \ln\left(\frac{w+d}{d}\right)\frac{\mathrm{d}I}{\mathrm{d}t},$$

where we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(a + bt) = b.$$

Therefore

$$\mathcal{E} = -\frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(1.00\ \mathrm{m})}{2\pi} \ln\left(\frac{1.10 \times 10^{-1}\ \mathrm{m}}{1.00 \times 10^{-2}\ \mathrm{m}}\right)(10.0\ \mathrm{A/s}) = -4.80 \times 10^{-6}\ \mathrm{V}.$$

(c) Since the emf is negative, and we have chosen $\mathrm{d}\mathbf{A}$ to point into the page, the current must point in the counterclockwise direction.