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(25 Pts)

1. An oscillating emf $\varepsilon(t) = \varepsilon_m \cos\omega t$ with amplitude $\varepsilon_m$ and frequency $\omega$ drives a current $I(t)$ through the circuit shown where R is a resistor, C is a capacitor, and L is an inductor.

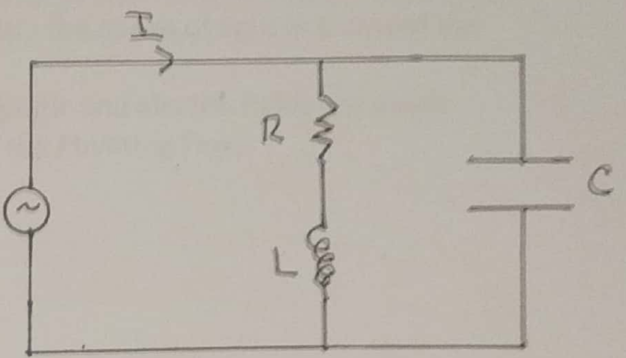
(8) a. Find the current $I_C$ that flows through the capacitor.

(12) b. Show that the current $I_{RL}$ that flows through the R-L branch is of the form

$$I_{RL}(t) = \frac{\varepsilon_m}{Z}\cos(\omega t - \phi)$$

where you must find $Z$ and $\phi$.

(5) c. Find the value of $\omega$ for which the total current $I = I_C + I_{RL}$ will be <u>in phase</u> with the driving emf. Will there always be such a frequency?




a) $\frac{q}{C} = \mathcal{E}_m \cos\omega t$

$q = C\mathcal{E}_m \cos\omega t$ 8

$I_C = \frac{dq}{dt} = -C\omega\mathcal{E}_m \sin\omega t$

b) $\mathcal{E}_m\cos(\omega t) - IR - L\frac{dI}{dt} = 0$

$IR + L\frac{dI}{dt} = \mathcal{E}_m\cos(\omega t) = Re(\mathcal{E}_m e^{i\omega t})$

$I = I_0 e^{i\omega t}$

$I_0 R + i\omega L I_0 = \mathcal{E}_m$

$I_0 = \frac{\mathcal{E}_m}{R + i\omega L} = \frac{\mathcal{E}_m(R - i\omega L)}{R^2 + (\omega L)^2} = \frac{\mathcal{E}_m(R + iX_L)}{R^2 + X_L^2}$

$I = Re(I_0 e^{i\omega t}) = \frac{\mathcal{E}_m}{\sqrt{R^2+X_L^2}} Re\left[\frac{(R - iX_L)(\cos\omega t + i\sin\omega t)}{\sqrt{R^2+(X_L)^2}}\right]$

$= \frac{\mathcal{E}_m}{\sqrt{R^2+X_L^2}} \frac{R\cos\omega t + X_L \sin\omega t}{\sqrt{R^2+(X_L)^2}}$ 12

Let $\cos\phi = \frac{R}{\sqrt{R^2+X_L^2}} \Rightarrow \sin\phi = \frac{X_L}{\sqrt{R^2+X_L^2}}$

So $I = \frac{\mathcal{E}_m}{\sqrt{R^2+X_L^2}}(\cos\omega t\cos\phi + \sin\omega t\sin\phi)$

Let $Z = \sqrt{R^2+X_L^2}$

$\Rightarrow I = \frac{\mathcal{E}_m}{Z}\cos(\omega t - \phi)$

c) $I = I_C + I_{RL} = \frac{\mathcal{E}_m}{X_C}$ 1

(25 Pts)

2. A plane electromagnetic wave in vacuum propagates in the y-direction, and has a magnetic field in the x-direction that is given by $B_x(y,t) = B_m \cos(ky - \omega t)$.

(9) a. Using the vector differential form of the Maxwell Equations, find the direction of the wave's electric field, and show that its magnitude is $E(y,t) = E_m \cos(ky - \omega t)$ where you must find $E_m$ in terms of $B_m$. Be sure to identify the speed of light in terms of the constants that appear in the Maxwell Equations.

(8) b. Prove that the energy densities of the wave's magnetic and electric fields are equal.

(8) c. Find the direction and the time-averaged value of the Poynting Flux.

a) $\nabla \times B = \cancel{\mu_0 \vec{J}}^{\,0} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

$\nabla \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & 0 & 0 \end{vmatrix} = \cancel{\frac{\partial B_x}{\partial z} \hat{j}}^{\,0} + -\frac{\partial B_x}{\partial y} \hat{k} = -(-B_m k \sin(ky - \omega t))$

$= B_m k \sin(ky - \omega t) \hat{k} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

So $E$ is in the $\hat{k}$, or z-direction

$\frac{\partial E}{\partial t} = \frac{B_m}{\mu_0 \varepsilon_0} k \sin(ky - \omega t)$

$\frac{1}{c}$ $\quad c^2$

$E = -\left(-\frac{1}{\omega} \frac{B_m}{\mu_0 \varepsilon_0} k \cos(ky - \omega t)\right) = \left(\frac{k}{\omega}\right)\left(\frac{1}{\mu_0 \varepsilon_0}\right) B_m \cos(ky - \omega t)$

$= \frac{1}{c} \cdot c^2 B_m \cos(ky - \omega t) = c B_m \cos(ky - \omega t)$

$E = E_m \cos(ky - \omega t)$

+9

b) $U_E = \frac{1}{2} \varepsilon_0 E^2 = \frac{1}{2} \varepsilon_0 (c B_m)^2 \cos^2(ky - \omega t) = \frac{1}{2} \varepsilon_0 \frac{1}{\mu_0 \varepsilon_0} B_m^2 \cos^2(ky - \omega t)$

$= \frac{1}{2\mu_0} (B_m \cos(ky - \omega t))^2 = \frac{B^2}{2\mu_0} = U_B$

+8

c) $S = \frac{E \times B}{\mu_0} = \frac{EB}{\mu_0} \hat{j} = \frac{c B_m \cos(ky - \omega t) B_m \cos(ky - \omega t)}{\mu_0} = \frac{c}{\mu_0} B_m^2 \cos^2(ky - \omega t)$

$\langle S \rangle = \frac{c}{\mu_0} B_m^2 \cdot \frac{1}{2} = \frac{c}{2\mu_0} B_m^2$

+8

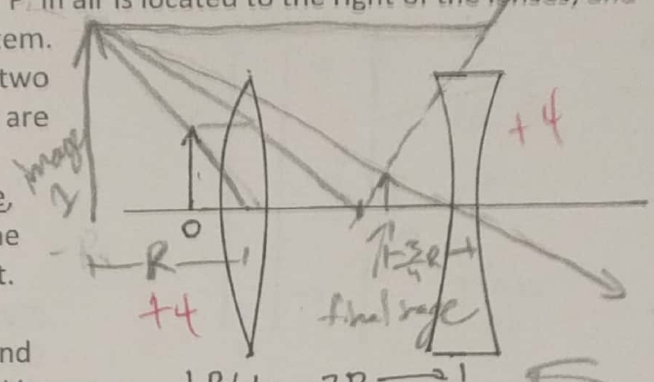
(25 Pts)

3. A thin bi-convex (double convex) lens and a thin bi-concave (double concave) lens are made from glass with an index of refraction $n_2 = 1.5$. The lenses have spherically curved surfaces with radii of curvature of <u>magnitude</u> R. The bi-convex lens is placed a distance 2R to the <u>left</u> of the bi-concave lens. An object O in air ($n_1 = 1.0$) is placed at a distance R/2 to the <u>left</u> of the bi-convex lens as shown. A person P in air is located to the right of the lenses, and observes the image formed by the system.

(5) a. Show that the focal lengths of the two lenses are equal in magnitude, but are of opposite sign.

(10) b. For the bi-convex lens <u>acting alone,</u> draw the principle rays and find the location of the image of the object. Is the image real or virtual?

(10) c. Now draw the principle rays and find the location of the image produced by the bi-concave lens that is observed by P. Is the image real or virtual? Upright or inverted?




P

Principle rays

$n_1 = 1$
$n_2 = 1.5$

a) For biconvex $\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] = \frac{n_2 - n_1}{n_1}\left[\frac{2}{R}\right]$

$\Rightarrow f = \frac{n_1 R}{2(n_2 - n_1)} = \frac{R}{2(.5)} = R$   +5

For biconcave $\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] = \frac{n_2 - n_1}{n_1}\left[-\frac{1}{R} - \frac{1}{R}\right] = \frac{n_2 - n_1}{n_1}\left[\frac{-2}{R}\right]$

$\Rightarrow f = \frac{-n_1 R}{2(n_2 - n_1)} = \frac{-R}{2(.5)} = -R$

So the focal lengths have equal magnitude but opposite sign

b) $\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$

$\frac{2}{R} + \frac{1}{s'} = \frac{1}{R}$

$\frac{1}{s'} = -\frac{1}{R}$

$s' = -R$

it's a virtual image +2 +4

R to the left of the biconvex lens

c) $\frac{1}{s} + \frac{1}{s'} = -\frac{1}{f}$

$\frac{1}{3R} + \frac{1}{s'} = -\frac{1}{R}$

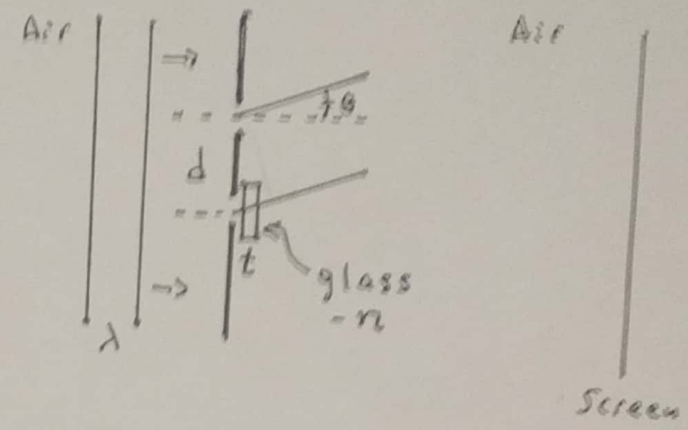
$\frac{1}{s'} = -\frac{4}{3R}$

$s' = \frac{3}{4}R$ ← $\frac{7}{4}R$ to the left of the biconcave lens +2

virtual image, upright +4

(25 Pts)

4. Two long, narrow slits are separated by a distance $d$ , and are illuminated by a coherent light source with wavelength $\lambda$ . The bottom slit is covered by a thin glass plate with thickness $t \ll d$ and index of refraction $n$ . In the direction specified by the angle $\theta$ , the slits illuminate a very distant screen as shown.




(8) a. Assuming that light goes through the glass plate at nearly normal incidence, prove that the phase difference between light from the top and bottom slits is given by

$$\Delta\phi = kd\sin\theta + kt(n-1)$$

(7) b. For fixed $\theta$ , prove that the minimum thickness $t_{min}$ for which a maximum that would have appeared in the absence of the glass plate is changed into a minimum is $t_{min} = \lambda/[2(n-1)]$.

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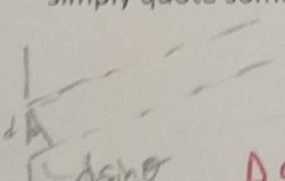
(10) c. For the value of $t_{min}$, prove that the intensity on the screen is given by

$$I = 4 I_0 \sin^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

where you must determine $I_0$. You must derive this result from first principles and not simply quote some memorized formula.

a)




$\Delta\phi > k_{glass}t - k_{air}t$

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$\delta = d\sin\theta$ without thin glass plate

$+ tn$ adding in glass plate

$-t \cdot 1$ subtracting out air

$\delta = d\sin\theta + tn - t = d\sin\theta + t(n-1)$

$\Delta\phi = k\delta = k(d\sin\theta + t(n-1)) = kd\sin\theta + kt(n-1)$

b) to change max into min, $\Delta\phi - \Delta\phi_{orig} = kd\sin\theta + kt(n-1) - kd\sin\theta$

$= kt(n-1) = (2m+1)\pi$ for some $m \geq 0$

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$\frac{2\pi}{\lambda} t(n-1) = (2m+1)\pi$

$t = \frac{(2m+1)\lambda}{2(n-1)}$

this is minimized when $m=0$, so

$t_{min} = \frac{\lambda}{2(n-1)}$

c) $E = E\cos(kr_1 - \omega t) + E(kr_2 - \omega t)$

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$= 2E\cos\left(k\frac{r_1+r_2}{2} - \omega t\right)\cos\left(k\frac{(r_1-r_2)}{2}\right)$

$= 2E\cos(kx - \omega t)\cos\left(\frac{\Delta\phi}{2}\right)$

$= 2E\cos(kx - \omega t)\cos\left(\frac{kd\sin\theta + k\frac{\lambda}{2(n-1)}(n-1)}{2}\right)$

$= 2E\cos(kx - \omega t)\cos(kd\sin\theta + \pi)$ $k = \frac{2\pi}{\lambda}$

$= 2E\cos(kx - \omega t)\sin(kd\sin\theta)$

(cont. on back)

c) (cont.)

$$I = \langle S \rangle = \langle \epsilon_0 E^2 \rangle = \epsilon_0 [2E\cos(kx-\omega t)\sin(kd\sin\theta)]^2$$

$$= 4 \cdot \tfrac{1}{2}\epsilon_0 E^2 \sin^2(kd\sin\theta)$$

$$= 4 I_0 \sin^2(kd\sin\theta)$$

$$= 4 I_0 \sin^2($$