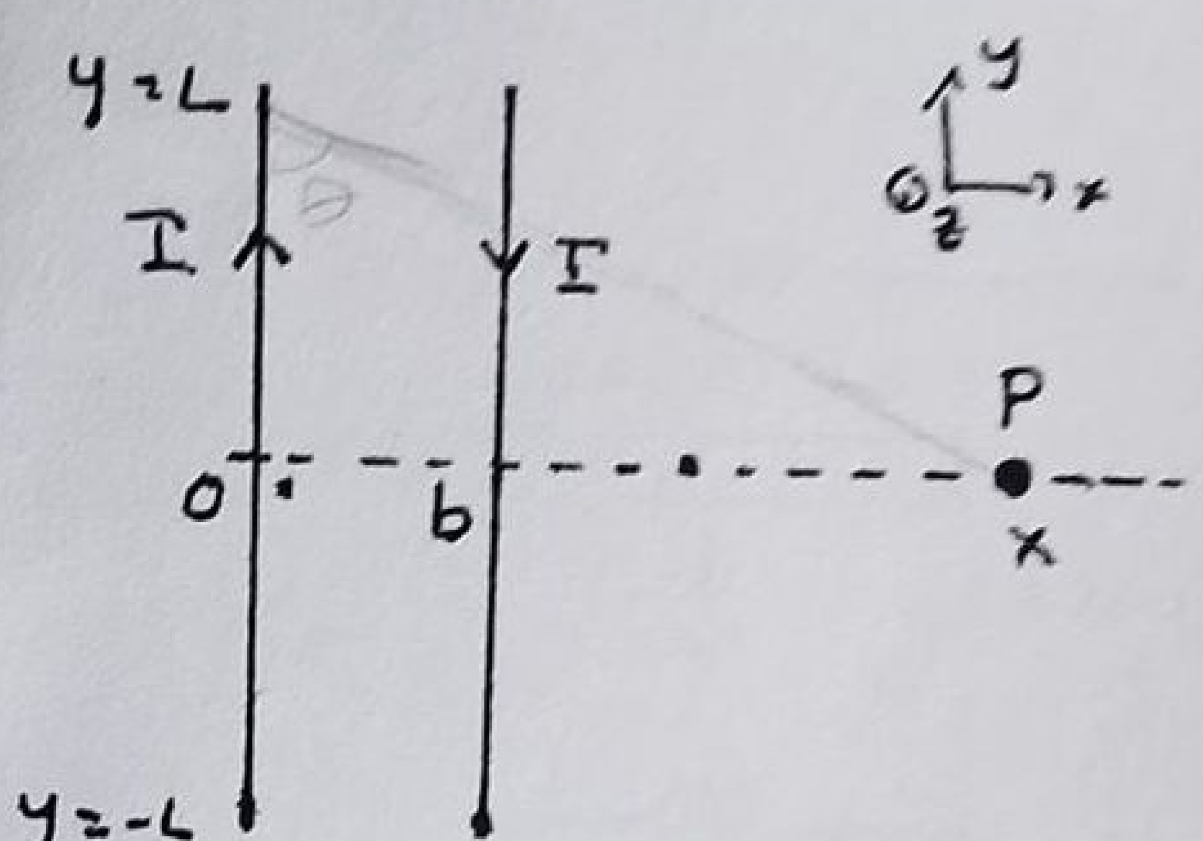
(35 Pts)

1. A thin wire at x = 0 runs from y = - L to y = + L, and carries a current I in the positive y-direction as shown. A second thin wire at x = b runs from y = - L to y = + L, and carries a current I in the negative y-direction as shown. Consider a point P located at x > b.

(15) a. Starting with the Biot-Savart Law, show that the magnetic field at P that is produced by the wire at x = 0 is



$$B_z = -\frac{\mu_0 I}{2\pi x}\frac{L}{(L^2+x^2)^{1/2}}$$

(5) b. Show that the total magnetic field B_{tot} at P from both wires is

$$B_{tot} = \frac{\mu_0 I}{2\pi}\left[\frac{L}{(x-b)[(x-b)^2+L^2]^{1/2}} - \frac{L}{x\,[L^2+x^2]^{1/2}}\right]$$

(5) c. If L >> x and b, and x >> b, prove that the total magnetic field is approximately given by

$$B_{tot} \approx \frac{\mu_0 I b}{2\pi x^2}$$

which represents a two-dimensional dipole field.

(10) d. Now suppose that x >> L and x >> b. First Taylor expand B_{tot} for x >> L and x - b >> L. Then Taylor expand B_{tot} for x >> b, and show that the total field is approximately

$$B_{tot} \approx \frac{\mu_0 I L b}{\pi x^3}$$

What does the $1/x^3$ term represent, and why is the result reasonable? -2

a) wire 1: $dB_1 = \frac{\mu_0}{4\pi}\frac{I\,dl\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\int\frac{I\,dy\sin\theta}{x^2+y^2}$ ✓

✓ $= \frac{\mu_0}{4\pi}\int_{-L}^{L}\frac{I\,dy\,x}{(x^2+y^2)^{3/2}} = \left.\frac{\mu_0 I x}{4\pi}\frac{y}{x^2(x^2+y^2)^{1/2}}\right]_{-L}^{L} = \frac{\mu_0 I}{4\pi x}\left(\frac{L}{\sqrt{x^2+L^2}}+\frac{L}{\sqrt{x^2+L^2}}\right)$

$= -\frac{\mu_0 I}{2\pi x}\frac{L}{(L^2+x^2)^{1/2}}$ ✓

b) $dB_2 = \frac{\mu_0}{4\pi}\int_{-L}^{L}\frac{I\,dy\sin\theta}{(x-b)^2+y^2} = \frac{\mu_0}{4\pi}\int_{-L}^{L}\frac{I\,dy\,(x-b)}{((x-b)^2+y^2)^{3/2}}$

$= \frac{\mu_0 I}{4\pi}\left(\frac{L}{(x-b)((x-b)^2+L^2)^{1/2}}+\frac{L}{(x-b)((x-b)^2+L^2)^{1/2}}\right)$ ✓

$$\boxed{B_{tot} = B_2 - B_1 = \frac{\mu_0 I}{2\pi}\left[\frac{L}{(x-b)((x-b)^2+L^2)^{1/2}} - \frac{L}{x(L^2+x^2)^{1/2}}\right]}$$ ✓

c) $L \gg x, b;\ x \gg b;\ B_{tot} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x-b}-\frac{1}{x}\right) = \frac{\mu_0 I}{2\pi}\left(\frac{b}{x^2-bx}\right) \approx \boxed{\frac{\mu_0 I}{2\pi}\left(\frac{b}{x^2}\right)}$ ✓

-2 d) $B_{tot} = \frac{\mu_0 I}{2\pi}\left[\frac{L}{\sqrt{1+\frac{(x-b)^2}{L^2}}} - \frac{L}{\sqrt{1+\frac{x^2}{L^2}}}\right] = \frac{\mu_0 I b}{2\pi}\left[\left(1+\frac{x-b}{2}-1-\frac{x}{2}\right)\right] = \frac{\mu_0 I}{2\pi}\left[\frac{2x}{x^4}\right] = \boxed{\frac{\mu_0 I L b}{\pi x^3}}$

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(35 Pts)

2. A vertical long thin wire carries a steady downward-directed current I_0 . A horizontal conducting rod of length a and mass m has one end at a distance b from the wire as shown, and moves vertically downward with a speed v .

(10) a. Use Ampere's Law to calculate the magnetic field produced by the wire at a distance r from the wire.

(10) b. Prove that the emf induced across the rod is given by

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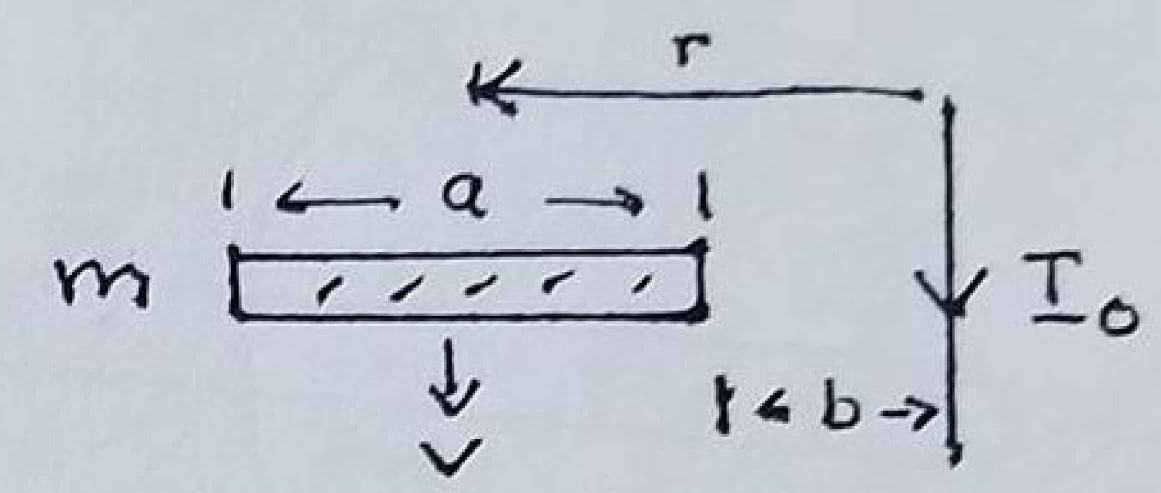
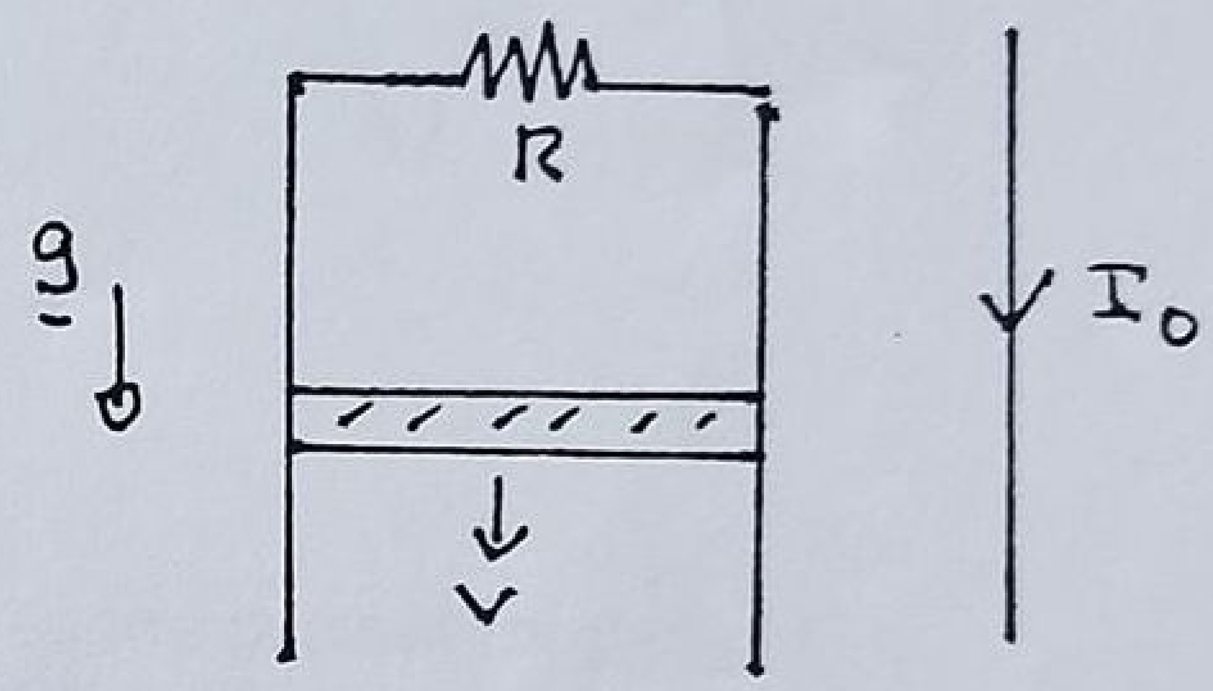
$$\varepsilon = -v \frac{\mu_0 I_0}{2\pi} \ln \left[\frac{b+a}{b}\right]$$



and explain the meaning of the minus sign.

(8) c. Now consider the rod to slide along frictionless vertical rails that close at the top through a resistor R. Find the direction and magnitude of the induced current that flows in the rail-rod system. Now find the vertical force on the rod due to the current.



(7) d. If at time t = 0 the rod starts to fall from rest under gravity g, show that the rod's speed is given by

$$v(t) = g\,\tau\,[1 - \exp(-t/\tau)] \qquad \tau = m\,R/\left[\frac{\mu_0 I_0}{2\pi} \ln\left(\frac{b+a}{b}\right)\right]^2$$

a) $\oint B\cdot dl = \mu_0 I$ loop: circle around axis

$B(2\pi r) = \mu_0 I$

$\boxed{B = \frac{\mu_0 I}{2\pi r}}$

b) $\mathcal{E} = (v\times B)\cdot dl$

$= vB\,dl$

$B = \int_b^{b+a} \frac{\mu_0 I_0\,dr}{2\pi r} = \frac{\mu_0 I_0}{2\pi} \ln r \Big]_b^{b+a}$

$= \frac{\mu_0 I}{2\pi} \ln\left(\frac{b+a}{b}\right)$

$\boxed{\mathcal{E} = -vB\,dl = -v\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{b}\right)}$

explain

-2

?

c) direction of current; counterclockwise

$I = \frac{\mathcal{E}}{R} = -\frac{v\mu_0 I_0}{2\pi R}\ln\left(\frac{b+a}{b}\right)$

$F = ILB = IBa$

$= -v\frac{\mu_0 I_0 \ln\left(\frac{b+a}{b}\right)}{2\pi R} \cdot \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{b}\right)$

$\boxed{F = -\frac{v}{R}\left(\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{b}\right)\right)^2}$

d) $F = mg - ma$

$= mg - m\frac{dv}{dt}$ $\quad m\frac{dv}{dt} - \frac{v}{R}\left(\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{b}\right)\right)^2$

$\frac{dv}{dt} - \frac{v}{mR}\left(\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{b}\right)\right)^2 = g$

$\mu(t) = e^{-\frac{1}{mR}\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{b}\right)^2}$

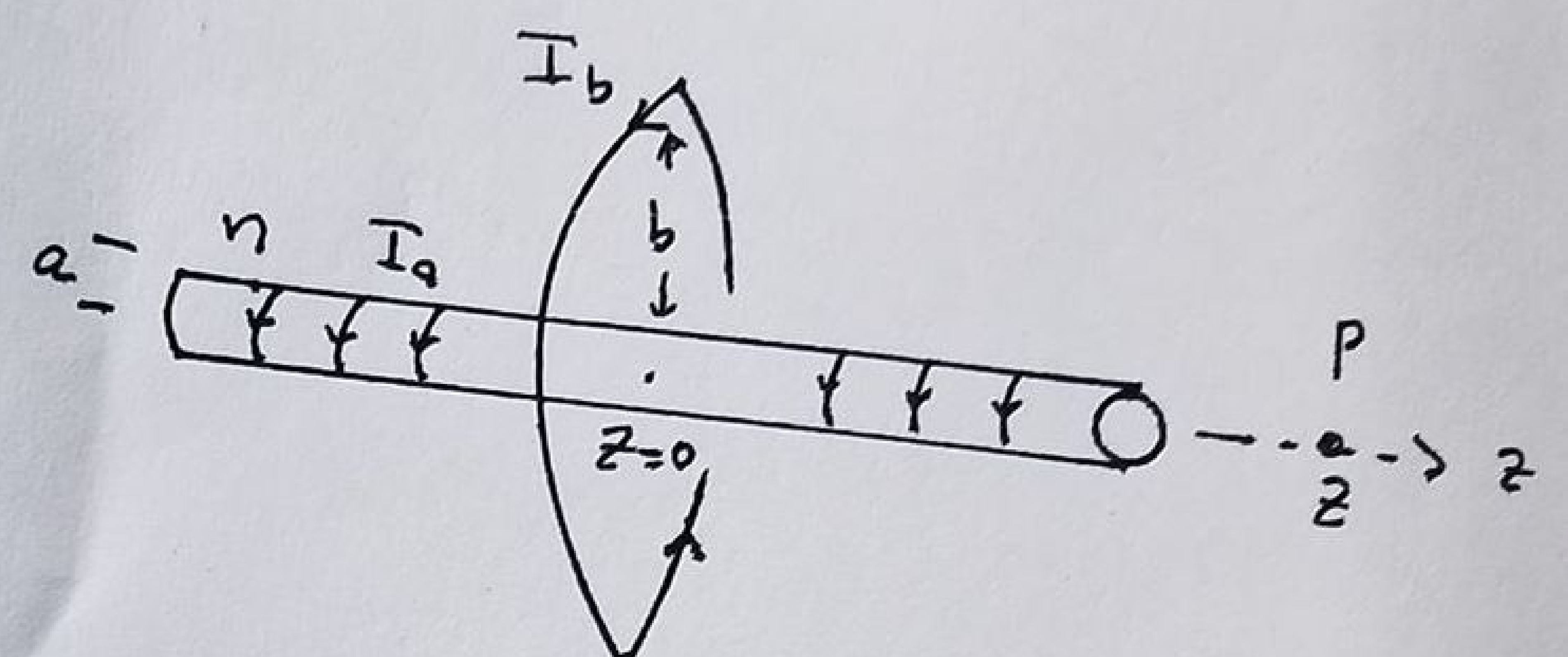
$v = \frac{g\,mR}{\left(\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{b}\right)\right)^2}\cdot\left(1 - e^{-t/\left(\frac{mR}{\left(\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{a}\right)\right)^2}\right)}\right)$

$\boxed{v = g\tau\left(1 - e^{-t/\tau}\right) \quad \tau = mR/\left(\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b+a}{a}\right)\right)^2}$

(30 Pts)

3. A very long, cylindrical (radius = a) solenoid is aligned along the z-axis, and has n turns of wire per unit length that each carry a current I_a in the direction shown. A circular loop (radius $b > a$) is centered at z = 0, is fixed in the x-y plane, and carries a current I_b in the direction shown.

(10) a. Use Ampere's Law to calculate the magnetic field produced inside the solenoid by the current windings of the solenoid, and find the solenoid's self-inductance per unit length.



(10) b. Use the Biot-Savart Law to prove that the magnetic field produced by the current loop at a distance z along the z-axis is

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$$B_b(z) = \frac{\mu_0}{2\pi}\frac{\pi b^2 I_b}{[z^2 + b^2]^{3/2}}$$

(10) c. Assume $a \ll b$ so that the magnetic field produced by the current loop $B_b(z)$ is essentially constant across the cross-sectional area of the solenoid. Now, by explicit calculation, prove that the mutual inductances M_{ab} and M_{ba} are both equal to $\mu_0 \pi a^2 n$. [Hint: You will have to integrate $B_b(z)$ over $-\infty < z < \infty$ to obtain the total flux from the current loop that threads through the solenoid.]

a) $\int B \cdot dl = \mu_0 I_{encl}$

$\int B \cdot dl = BL = \mu_0 I_{enc} = \mu_0 I_a n L$

10 $BL = \mu_0 I_a n L$

$\boxed{B = \mu_0 I n}$

$L = \dfrac{n\Phi}{I}$

$\Phi = BA = B\pi a^2$

$L = \dfrac{n\mu_0 I n \pi a^2}{I}$

$\boxed{L = \mu_0 n^2 \pi a^2}$

b) $dB = \dfrac{\mu_0}{4\pi}\dfrac{I dl \times \hat{r}}{r^2} = \dfrac{\mu_0}{4\pi}\dfrac{I dl}{r^2}$

10

$r^2 = z^2 + b^2$

$dB_z = dB\cos\theta = \dfrac{\mu_0 I (2\pi b)}{4\pi (z^2+b^2)}\dfrac{b}{\sqrt{z^2+b^2}}$

$\boxed{B_b(z) = \dfrac{\mu_0 I_b \pi b^2}{2\pi (z^2+b^2)^{3/2}}}$

c) $M_{ab} = \dfrac{N_a \Phi_{ab}}{I_b} = \dfrac{\pi a^2}{b^2}\Phi_{ab} = \int_{-\infty}^{\infty} B_b(z) = \dfrac{a^2 n}{b^2}\int_{-\infty}^{\infty} \dfrac{\mu_0 I_b \pi b^2 n}{2\pi (z^2+b^2)^{3/2}} dz$

10

$= \dfrac{n\mu_0 I_b a^2 \pi}{I_b}$

$= \boxed{\mu_0 n \pi a^2}$

$= \dfrac{a^2 n}{b^2}\dfrac{\mu_0 I \pi b^2}{2\pi}\left[\dfrac{z}{b^2(z^2+b^2)^{1/2}}\right]_{-\infty}^{\infty}$

$= \dfrac{\mu_0 I_b \pi b^2}{2\pi}(2)\cdot\dfrac{a^2 n}{b^2}$

$= \mu_0 I_b b^2 \cdot \dfrac{a^2}{b^2}$

$= \mu_0 I_b a^2 n$

$\boxed{\begin{aligned} M_{ba} &= M_{ab} \\ &= \mu_0 n \pi a^2 \end{aligned}}$