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(25 Pts)

1. Four very long wires, each carrying a current of magnitude I , are arranged in a square pattern with the currents in the directions shown. The diagonal of the square is of length 2a. Point P is a distance x from the center of the square (x = 0, y = 0).

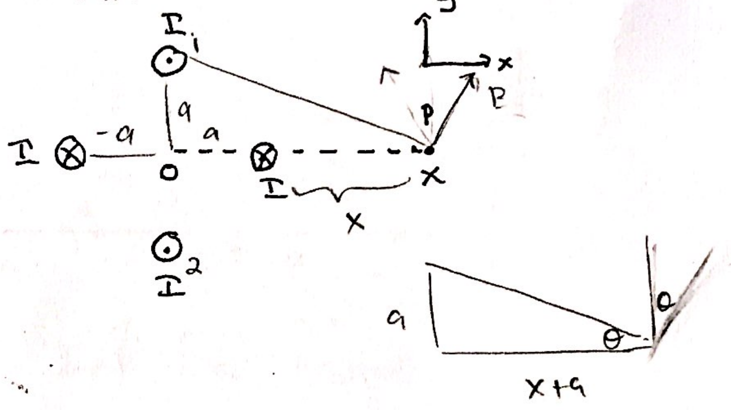
(10) a. Find the direction and magnitude of the magnetic field produced by the two wires located on the x-axis at x = - a and x = +a .

(10) b. Find the direction and magnitude of the magnetic field produced by the two wires located on the y-axis at y = - a and y = +a .

(5) c. Show that for x >> a, the magnitude of the total magnetic field at P is approximately given by

$$|\underline{B}| \approx \frac{2\mu_0 I a^2}{\pi x^3}$$

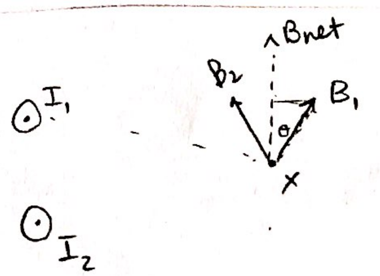
$B_{wire} = \frac{M_0 I}{2\pi r}$ $\quad d\vec{B} = \frac{M_0 I}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2}$



a) $\vec{B}$ is in the -y-direction $(-\hat{j})$ +2

$$B = \frac{M_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{2a+x}\right] = \frac{M_0 I}{2\pi}\left[\frac{2a+x+x}{x(2a+x)}\right] = \frac{M_0 I (a+x)}{\pi x (2a+x)}$$

b) $\vec{B}$ is in the +y-direction $(\hat{j})$ +2



$B_{net} = 2B\cos\theta$

$$B = \frac{M_0 I}{2\pi} \cdot 2 \cdot \cos\theta, \quad \cos\theta = \frac{x+a}{\sqrt{(x+a)^2+a^2}}$$

+3

$$B = \frac{M_0 I}{\pi} \cdot \frac{x+a}{\sqrt{(x+a)^2+a^2}}$$

c) total field: (+y-direction)

$$B = \frac{M_0 I}{\pi}\left[\frac{x+a}{\sqrt{(x+a)^2+a^2}} - \frac{a+x}{x(2a+x)}\right]$$

$$x \gg a \quad \to \quad \frac{x}{x\sqrt{1+\frac{a^2}{x^2}}} - \frac{x}{x^2\left(1+\frac{2a}{x}\right)}$$

$$\to B = \frac{M_0 I}{\pi}\left[\left(1-\frac{a}{x}\right) - \quad \right]$$

(20 Pts)

2. A thin wire runs along the x-axis from x = a to x = L, and carries a current I in the positive x-direction. A second thin wire runs along the x-axis from x = - L to x = - a, and also carries a current I in the positive x-direction. Consider a Point P located at (x = 0, y).

(15) a. Find the direction of the magnetic field at Point P, and show that its magnitude is given by

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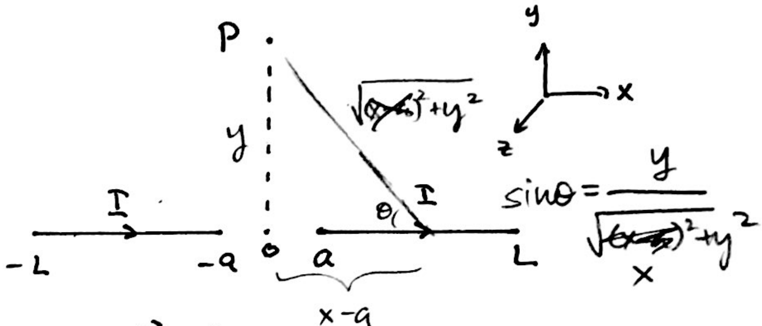
$$|\underline{\mathbf{B}}| = \frac{\mu_0 I}{2\pi y}\left[\frac{L}{(L^2+y^2)^{\frac{1}{2}}} - \frac{a}{(a^2+y^2)^{\frac{1}{2}}}\right]$$



(5) b. If $L \to \infty$ and y << a, show that

$$|\underline{\mathbf{B}}| \approx \frac{\mu_0 I y}{4\pi a^2}$$

Why should the $|\underline{\mathbf{B}}|$ vanish at the origin?

$$I d\vec{l} \times \hat{r} = I l \sin\theta$$

a) direction in +z : $(\hat{k})$

right wire: $\frac{1}{2}B = \frac{\mu_0 I}{4\pi}\int_a^L \frac{y}{(x^2+y^2)^{3/2}}\,dx = \frac{\mu_0 I y}{4\pi}\left[\frac{x}{y^2(x^2+y^2)^{1/2}}\Big|_a^L\right]$

15 $= \frac{\mu_0 I}{4\pi y}\left[\frac{L}{(L^2+y^2)^{1/2}} - \frac{a}{(a^2+y^2)^{1/2}}\right]$

So $B = \frac{2\mu_0 I}{4\pi y}\left[\frac{L}{(L^2+y^2)^{1/2}} - \frac{a}{(a^2+y^2)^{1/2}}\right] \stackrel{\checkmark}{=} \frac{\mu_0 I}{2\pi y}\left[\frac{L}{(L^2+y^2)^{1/2}} - \frac{a}{(a^2+y^2)^{1/2}}\right]$

b) $L \to \infty$, $y \ll a$

$$\frac{1}{(1+x)^{1/2}} = 1 - x/2$$

5 $B = \frac{\mu_0 I}{2\pi y}\left[\frac{L}{(L^2)^{1/2}} - \frac{a}{a(1+\frac{y^2}{a^2})^{1/2}}\right]$

$= \frac{\mu_0 I}{2\pi y}\left[\frac{y^2}{2a^2} - 1 + 1\right] \stackrel{\checkmark}{=} \frac{\mu_0 I y}{4\pi a^2}$

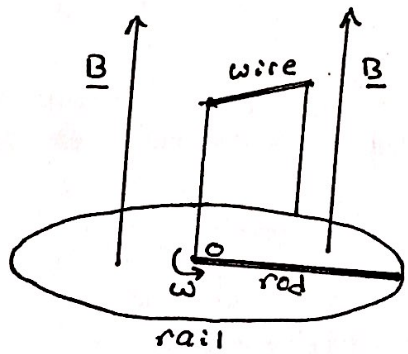
B should vanish at the origin because there appears to be 2 infinite wires so that $I d\vec{l} \times \hat{r} = 0$ since $\theta = 0$.

(30 Pts)

3. A thin conducting rod of length L and mass m rotates in the x-y plane about the z-axis through one end of the rod (the origin) with an angular frequency ω as shown. A uniform and temporally constant magnetic field **B** is parallel to the z-axis. The rod slides without friction along a conducting, circular, fixed rail that is connected by a fixed wire to the origin as shown, thereby allowing a current to flow along the rod and the rail, and to close back to the rod through the wire. The system has a total resistance R to the flow of current.

(10) a. Find the direction and magnitude of the emf that is induced by the rotating rod.

(5) b. Find the current that flows in the circuit.



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Recall that the moment of inertia of the rod about the z- axis is $m\,L^2/3$, and that the differential torque is $\underline{d\tau} = \underline{r} \times \underline{dF}$.

(10) c. At time t = 0, the angular frequency of the rod is ω_0. Show that $\omega\,(t)$ is

$$\omega\,(t) = \omega_0 \exp\,[-\,t/\tau]$$

where you must determine the characteristic damping time τ.

(5) d. By integrating $d\phi/dt = \omega\,(t)$, show that the rod will not make a complete rotation if $\omega_0 < 2\pi/\tau$.

a) $\mathcal{E} = \oint \vec{v} \times \vec{B} \cdot d\vec{l}$ $\quad \vec{v} = r\vec{\omega}$ $\quad$ r is radius $\quad$ 10

$\mathcal{E} = \int_0^r B\omega r' dr' = \frac{1}{2} B\omega r^2 = \frac{1}{2} B\omega r^2$, points radially out

b) $\mathcal{E} = IR$ $\quad I = \dfrac{\frac{1}{2} B\omega r^2}{R}$ $\quad$ 5

c) $I_{inert} = \frac{1}{3} mL^2$ $\quad \vec{\tau} = \vec{r} \times \vec{F}$ $\quad \perp$

$d\vec{F} = I d\vec{l} \times \vec{B}$ $\quad \vec{\tau} = \vec{r} \times (I d\vec{l} \times \vec{B})$ $\quad \omega(t) = \omega_0 e^{-t/\tau}$

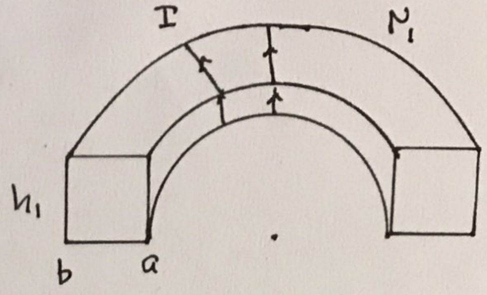
$\vec{\tau} = I\alpha$ where $\alpha = \omega'$ $\quad \frac{1}{3} mL^2 \alpha = \vec{r} \times (I d\vec{l} \times \vec{B})$ current

so $\omega(t) = \int \alpha(t)$

0

(25 Pts)

4. A toroidal solenoid with an inner radius a, an outer radius b, and a height h_1 has N_1 turns that carry a current I_1.



(12) a. Use Ampere's Law to find the magnetic field inside the solenoid, and show that the self-inductance is

$$L_1 = \frac{\mu_0 N_1^2 h_1}{2\pi} \ln (b/a)$$

A second toroidal solenoid with inner radius c ($< a$), outer radius d ($> b$), height h_2 ($> h_1$), and N_2 turns that carry a current I_2 concentrically surrounds the first solenoid as shown in cross-section.

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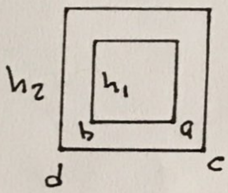

(13) b. By explicit calculation, show that the mutual inductances M_{21} and M_{12} are equal. Why are M_{21} and M_{21} not equal to $(L_1 L_2)^{1/2}$?

a) 6 $B_{\text{toroidal solenoid}} = \frac{\mu_0 N I}{2\pi r}$

Φ

$\cancel{B} = \frac{\mu_0 N I}{2\pi} \int_a^b \frac{h_1 dr}{r} = \frac{\mu_0 N_1 I}{2\pi} \ln(b/a)$

$L_1 = \frac{N_1 \Phi}{I} = \frac{N_1 \mu_0 N_1 \cancel{I} \ln(b/a) h_1}{2\pi \cancel{I}}$

$\Phi = BA$
barea

$= \frac{\mu_0 N_1^2 h_1}{2\pi} \ln(b/a)$

b) 10

$M_{21} = \frac{N_2 \Phi_{21}}{I_1} = M_{12} = \frac{N_1 \Phi_{12}}{I_2}$

$= \frac{N_2 \mu_0 I_1 N_1 \ln(b/a) h_1}{I_1} \stackrel{?}{=} \frac{N_1 \mu_0 N_2 I_2 \ln(b/a) h_1}{I_2}$

$\mu_0 N_1 N_2 \ln(b/a) h_1$

$\stackrel{\checkmark}{=} \mu_0 N_1 N_2 \ln(b/a) h_1$

L_1 depends on $\ln(b/a)\, h_1$

L_2 depends on $\ln(d/c)\, h_2$

toroidal solenoids aren't the same geometry → magnetic field lines leak.