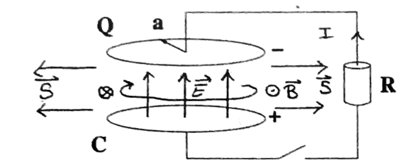
16



$I = -dq/dt$

$\frac{q}{C} - IR = 0$

$\int \frac{dt}{RC} = \int \frac{dq}{q}$

$q = Qe^{-t/RC}$

$E = \frac{\sigma}{\epsilon_0} = \frac{q}{\epsilon_0 A}$

1) A parallel-plate capacitor of capacitance C constructed with circular plates of radius a is given an initial charge Q and connected in series across an open switch and an ohmic resistor of resistance R. The switch is closed at $t = 0$.

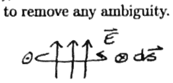
- 1a) (5 points) Write or derive the charge on the capacitor, the current through the resistor and the electric field within the capacitor as functions of time. Sketch the electric field on the diagram above, assuming the charge polarity is as shown. For full credit, use ⊙ and ⊗ if/as needed to remove any ambiguity.

$$\boxed{\begin{aligned} q &= Qe^{-t/RC} \\ I &= \frac{Q}{RC} e^{-t/RC} \\ E &= \frac{Q}{\epsilon_0 \pi a^2} e^{-t/RC} \end{aligned}}$$

2

- 1b) (10 points) Find the magnetic field inside the capacitor as a function of radial distance from the symmetry axis of the capacitor (and time), and sketch the field on the diagram above. For full credit, use ⊙ and ⊗ if/as needed to remove any ambiguity.

$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$



$B 2\pi r = \mu_0 \epsilon_0 \frac{d}{dt} \left[\frac{Q}{\epsilon_0 \pi a^2} e^{-t/RC} \cdot \pi r^2 \right]$

$$\boxed{B = -\frac{\mu_0 Q r}{2\pi a^2 RC} e^{-t/RC}}$$

opposite ds!

6

- 1c) (10 points) Find the magnitude of the Poynting vector at the relevant boundaries of the capacitor and sketch it on the diagram on the previous page. For full credit, use ⊙ and ⊗ if/as needed to remove any ambiguity.

$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ ⇒ in this case, directed outward as shown...

$S(a) = \frac{1}{\mu_0} E(a) B(a)$

$S(a) = \frac{1}{\mu_0} \frac{Q}{\epsilon_0 \pi a^2} e^{-t/RC} \frac{\mu_0 Q}{2\pi a RC} e^{-t/RC}$

$$\boxed{S(a) = \frac{Q^2}{\epsilon_0 2\pi^2 a^3 RC} e^{-2t/RC}}$$

6

- 1d) (5 points) Show that the rate at which electromagnetic waves carry energy out of the capacitor is the same as the rate at which the resistor consumes the energy in the circuit. $d \equiv$ plate separation ⇒ $C = \frac{\epsilon_0 \pi a^2}{d}$

$P = \int \vec{S} \cdot d\vec{A}$

$P = \int |\vec{S}| |dA| \cos(0)$

$P = S(a) 2\pi a d$

$P = \frac{Q^2 d}{\epsilon_0 \pi a^2 RC} e^{-2t/RC}$

$P = \frac{RC}{\frac{\epsilon_0 \pi a^2}{d}} \left(\frac{Q}{RC} e^{-t/RC} \right)^2$

$P = \frac{RC}{C} I^2$ ⟹ $\boxed{P = I^2 R}$ ✓

2

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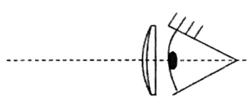
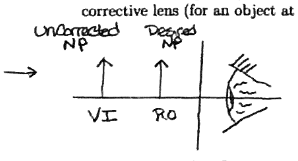


2) As we age, our true-lenses lose their plasticity. This leads to "far-sightedness", a condition in which the lens of the eye focuses light from near objects behind (rather than on) the retina. Symptomatically, age-related far-sightedness is characterized by a receding near-point (the distance from your eye to the closest point on which you can comfortably focus) - people of my age often have to hold printed material out at arm's-length when we forget our reading glasses.

- 2a) (5 points) The simplest way to deal with this condition is to add a corrective lens near the eye that maps objects located at the desired near-point to the location of the far-sighted individual's uncorrected near-point. Draw a quick diagram that shows the relative locations of the eye, the corrective lens, the uncorrected near-point and the desired near-point (make sure you label each!). Should the image produced by the corrective lens (for an object at the desired near-point) be real or virtual? Explain.



We need to produce an image at the uncorrected NP (where we can still focus) - its on the opposite side to the emerging light

$\boxed{\text{VIRTUAL IMAGE}}$

0

- 2b) (10 points) Call the distance between the lens of the eye and the **uncorrected** near-point $|P_{np}|$. Call the distance between the lens of the eye and the **desired** near-point d. Assume the corrective lens is adjacent to the eye and find the focal length of the lens that will correct the condition our subject suffers from. Be explicit with the sign (that is, make sure the grader knows whether the power is positive or negative). You may guess at the sign, but you'll get more credit if it is justified by the calculation. Is the corrective lens diverging or converging? Is that what you expect? (Why?)

$P = d \quad q = -|P_{np}|$

$\frac{1}{d} - \frac{1}{|P_{np}|} = \frac{1}{f}$

$$\boxed{f = \frac{d\,|P_{np}|}{|P_{np}| - d}}$$

$|P_{np}| > d$; $f > 0$ ⇒ Converging lens. For far-sighted folk, the true lens is not converging enough (distant objects focus behind retina), so adding a converging lens helps!

8

- 2c) (5 points) My current near-point is about 50 cm, but my reading classes give me an effective near-point of 10 cm. What is the focal length for my corrective lenses? What is the magnification associated with an object placed at the desired near-point?

$f = \frac{(10\,cm)(50\,cm)}{40\,cm}$ $\quad M = -\frac{q}{p} = +\frac{50\,cm}{10\,cm}$

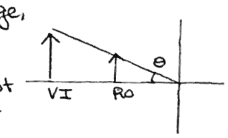
$$\boxed{\begin{aligned} f &= 12.5\,cm \\ M &= 5 \end{aligned}}$$

5

- 2d) (5 points) Interestingly enough, I don't notice any magnification when I look through my reading classes. My optometrist has, in fact, told me that a well-tuned pair of glasses will not magnify print. Explain the consistency between this observation and the results from part c.

We're looking at a bigger image, farther away...

The angular size (which is what the brain responds to) is the same...



5

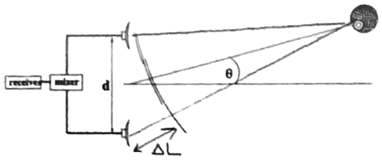
- 2e) (5 points) Reading glasses are not actually placed adjacent to the eye - there's some distance from lens-to-lens. Will the focal length have to increase or decrease to compensate for this? Explain.

d and $|P_{np}|$ both decrease by the same amount... $|P_{np}| - d$ won't change, $d\,|P_{np}|$ gets smaller...

$\boxed{\text{The focal length has to decrease}}$

5

4



3) In the illustration above, one sees a pair of radio-telescope antennae separated by a distance d. The antennae are tuned to monitor signals that have a wavelength λ.

- 3a) (5 points) State Rayleigh's criterion (qualitatively).

The first-order minimum associated with one object overlaps the central maximum of another when the objects are barely resolvable.

0

- 3b) (5 points) Find the angular separation between a pair of distant radio sources that are barely-resolvable in this radio-telescope.

Where is the first-order min?

$\Delta\theta_{path} = (2(0)+1)\pi$

$K d \sin\theta = \pi$

$\frac{2\pi}{\lambda} d \sin\theta = \pi$

hopefully θ is small

$\boxed{\theta_{BR} \sim \frac{\lambda}{2d}}$

1

- 3c) (5 points) Using your answer to part b, explain the advantage one might get by linking together antennae separated by large distances [this is the principle on which the Very Large Baseline Antenna (VLBA) operates].

A large baseline (d) means we can resolve to ever smaller angular separation – we can see more detail.

0

43/90

- 3d) (10 points) Radio signals from a distant source arrive at the antennae slightly out of phase (say, by an amount $\Delta\Phi$). If we draw one line from the midpoint of the antennae to the distant object, and another line along the bisector between the antenna, what are the possible values of the angle between the lines (θ)?

$\Delta\theta_{path} = \Delta\Phi + 2\pi N$ ← 2π ambiguity

$\frac{2\pi}{\lambda} d \sin\theta = \Delta\Phi + 2N\pi$

$\boxed{\sin\theta = \frac{\lambda}{d}\left(\frac{\Delta\Phi}{2\pi} + N\right)}$

2

- 3e) (5 points) Signals are taken from each antenna, by identical cables, to a mixer, where they are combined by simple superposition. The combined signal is then sent to a receiver for processing. Explain how one might add a time-delay to the signal in one of the cables to optimize the combined signal and find the amount of time-delay required.

Part d suggests we can 'point' the antenna by using phase difference.

$\Delta\theta_{TOT,\,mixer} = \Delta\theta_{line\,1} - \Delta\theta_{line\,2} = 2\pi N$ (maximas) ← TAKE N=0

$\underset{\text{desired direction}}{\Delta\Phi} - \underset{\text{time induced phase delay}}{\omega\Delta t} = 0$

$2\pi f \Delta t = \Delta\Phi$

$\frac{2\pi c}{\lambda}\Delta t = \Delta\Phi$

$\boxed{\Delta t = \frac{\Delta\Phi\,\lambda}{2\pi c}}$

1