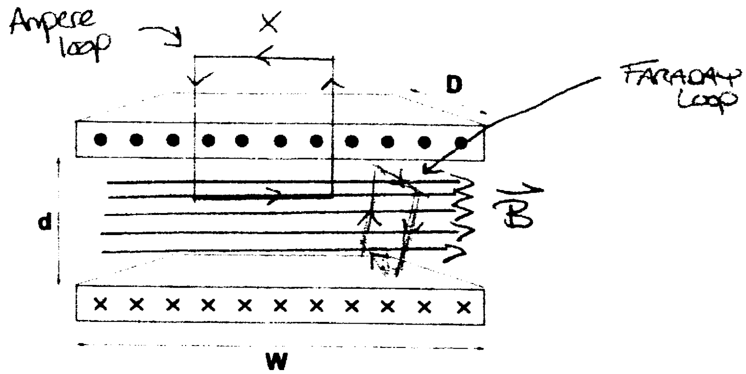
Ampere's LAW:

$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$

$Bx = \mu_0 K x$

$B = \mu_0 K$



1) Two thin, parallel, conducting sheets of dimension $D \times W$ are separated by a distance d as shown ($D >> W >> d$). The sheets each carry a *linear* current density K, one into the plane of the page, one out, as shown.

- 1a) (15 points) Use Faraday's law to calculate the self-inductance of the arrangement.

The Faraday loops will actually run perpendicular to the lines of $\vec{B}$. Take $d\vec{A}$ parallel to $\vec{B}$

$\Phi_B = \int \vec{B}\cdot dA$

$\Phi_B = \mu_0 K D d$

$K = I/W$

$\Phi_B = \frac{\mu_0 I D d}{W}$

$\mathcal{E}_i = -\frac{d\Phi_B}{dt}$

$\mathcal{E}_i = -\frac{\mu_0 d D}{W}\frac{dI}{dt}$

$L = \left|\frac{\mathcal{E}_i}{dI/dt}\right| \Longrightarrow \boxed{L = \frac{\mu_0 d D}{W}}$

- 1b) (10 pts) Verify your answer to the first part using energy considerations.

$u_B = \frac{1}{2\mu_0}B^2$

$u_B = \frac{1}{2}\frac{\mu_0}{W^2}I^2$

$U = \int u_B dV$

$U = \frac{1}{2}\frac{\mu_0}{W^2}I^2 dDW$

$U = \frac{1}{2}\frac{\mu_0 dD}{W}I^2 \longleftrightarrow$ Compare to $U = \frac{1}{2}LI^2$

$$\boxed{L = \frac{\mu_0 d D}{W}}$$

- 1c) (5 pts) Define $\hat{L}$ as the inductance per unit length (measured along the current) and $\hat{C}$ as the capacitance per unit length. Calculate $\frac{1}{\sqrt{\hat{L}\hat{C}}}$. This quantity plays an important role in the practical evaluation of transmission lines. Care to guess what it is?

$$\hat{L} = \frac{\mu_0 d}{W} \qquad \hat{C} = \frac{\epsilon_0 W}{d}$$

$$\boxed{\frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

Essentially a fundamental Constant.

... with dimensionality $\frac{L}{T}$

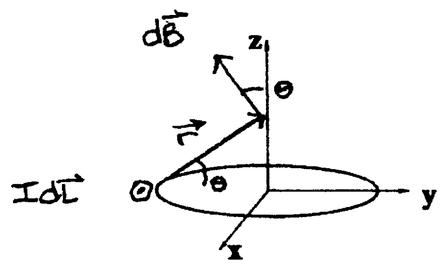
... (the speed of light in Vaccum)

Symmetry:

$\vec{B} = B_z \hat{z}$

$B_z = \int dB_z$

$dB_z = dB \cos\theta$



$\cos\theta = \frac{R}{\sqrt{R^2+z^2}}$

- 2a) (10 points) A circular, conducting loop of radius R lies in the x,y-plane, centered on the origin. A current I flows through the loop such that at $x = +R$, the current is headed in the $+\hat{y}$ direction, and at $x = -R$ the current is headed in the $-\hat{y}$ direction. Derive the resultant magnetic field (magnitude and direction) at every point along the z-axis.

$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{L} \times \hat{r}}{r^2}$

$dB = \frac{\mu_0 I}{4\pi} \frac{|d\vec{L} \times \hat{r}|}{r^2}$

$dB = \frac{\mu_0 I}{4\pi} \frac{dL}{(R^2+z^2)}$

$dB_z = dB \cos\theta$

$dB_z = \frac{\mu_0 I R\, dL}{4\pi (R^2+z^2)^{3/2}}$

$B_z = \frac{\mu_0 I R}{4\pi (R^2+z^2)^{3/2}} \int dL$

$$\boxed{\vec{B} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \hat{z}}$$

- 2b) (10 points) Let's replace the loop with an infinitesimally-thin, uniform ring of electric charge that extends from r to $r + dr$. If the surface charge-density on the ring is given by σ and the ring rotates about the z-axis with a constant angular velocity $\vec{\omega}$ (recall, the direction of $\hat{\omega}$ is obtained by using the right-hand rule with the physical motion of points on the ring) find the magnitude and direction of the (infinitesimal) magnetic field produced at every point along the z-axis.

Assume $\hat{\omega} = \hat{z} \Rightarrow$ then our ring is well-described by the work we did in part a

$d\vec{B} = \frac{\mu_0 dI\, r^2}{2(r^2+z^2)^{3/2}} \hat{\omega} \leftarrow dI = \frac{dq}{T} = \frac{\sigma dA}{2\pi/\omega} = \sigma \omega r dr$

$d\vec{B} = \frac{\mu_0 \sigma \omega r^3 dr}{2(r^2+z^2)^{3/2}} \hat{\omega}$

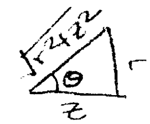
$$\boxed{d\vec{B} = \frac{\mu_0 \vec{\omega}}{2} \cdot \frac{\sigma r^3 dr}{(r^2+z^2)^{3/2}}}$$

- 2c) (10 points) Now lets replace the thin ring of part b with a washer that extends from $r = a$ to $r = b$. Charge is distributed over the washer with a surface charge density

$$\sigma(r) = \frac{q\,ab}{2\pi\,(b-a)}\,\frac{1}{r^3}$$

and it rotates with a constant angular velocity $\vec{\omega}$. Find the magnitude and direction of the magnetic field produced at every point on the z-axis.

Superpose an infinite number of infinitesimal rings ∴



$$d\vec{B} = \frac{\mu_0 \vec{\omega}}{2}\,\frac{q\,ab}{2\pi(b-a)}\,\frac{dr}{(r^2+z^2)^{3/2}}$$

$r = z\tan\theta$

$dr = \frac{z\,d\theta}{\cos^2\theta}$

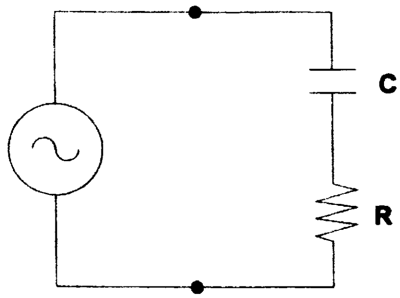
$\sqrt{r^2+z^2} = \frac{z}{\cos\theta}$

$$d\vec{B} = \frac{\mu_0 \vec{\omega}\, q\,ab}{4\pi(b-a)}\,\frac{z\,d\theta}{\cos^2\theta}\,\frac{\cos^3\theta}{z^3}$$

$$\int d\vec{B} = \frac{\mu_0 \vec{\omega}\, q\,ab}{4\pi(b-a)\,z^2}\int d\theta\,\cos\theta$$

$$\vec{B} = \frac{\mu_0\, q\,ab\,\vec{\omega}}{4\pi(b-a)\,z^2}\,\sin\theta\,\Big|_{\theta_a}^{\theta_b}$$

$$\boxed{\vec{B} = \frac{\mu_0\, q\,ab\,\vec{\omega}}{4\pi(b-a)\,z^2}\left[\frac{b}{\sqrt{b^2+z^2}} - \frac{a}{\sqrt{a^2+z^2}}\right]}$$

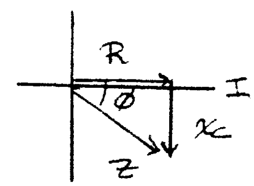


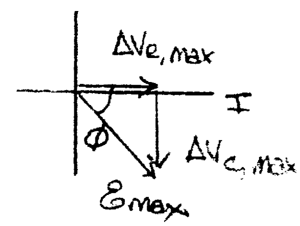
3) An capacitor (C) and a resistor (R) are connected in series across a source of alternating EMF ($\xi(t) = \xi_{max}\cos(\omega t)$). You may do the following calculations in complex space, but keep in mind, the final answers must all be real.

- 3a) (10 points) What is the impedance of RC combination? What is the amplitude of the current that passes through it? Does the current through this impedance lead or lag the voltage across it? By how much?

- $Z = \sqrt{(1/\omega C)^2 + R^2}$
- $I_{max} = \frac{\mathcal{E}_{max}}{Z} = \frac{\mathcal{E}_{max}}{\sqrt{(1/\omega C)^2 + R^2}}$
- Current leads Voltage by $\phi = \tan^{-1}\left(\frac{1}{\omega CR}\right)$

Series





- 3b) (10 points) What is the voltage amplitude across the resistor? What is the voltage amplitude across the capacitor? Will the sum of these voltage amplitudes equal the voltage amplitude of the source? Explain.

$$\Delta V_{R,max} = I_{max} R = \mathcal{E}_{max} \frac{R}{\sqrt{(1/\omega C)^2 + R^2}}$$

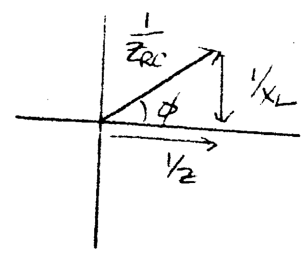
$$\Delta V_{C,max} = I_{max} X_C = \mathcal{E}_{max} \frac{1/\omega C}{\sqrt{(1/\omega C)^2 + R^2}}$$

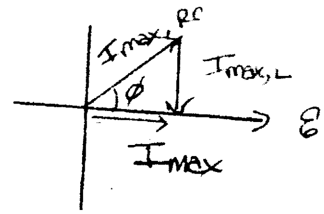
$$\mathcal{E}_{max} \neq \Delta V_{R,max} + \Delta V_{C,max}$$

Phase differences mean the voltage across the resistor and inductor peak at different moments in time.

• 3c) (10 points) For a lot of good reasons, it is often desirable to present the source with a purely resistive load. We can tune-out the reactance in the RC network by adding an inductor in parallel with it (across the two dots in the circuit). What value should the inductor have? What will be the value of the new impedance seen by the source? Will current through this new LRC combination lead or lag the voltage across it? By how much?

Parallel





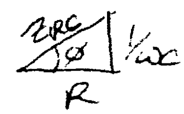
We can solve this geometrically:

$$\frac{1}{X_L} = \frac{1}{Z_{RC}} \sin\phi$$

$$X_L = Z_{RC}^2\, \omega C \qquad \leftarrow X_L = \omega L$$

$$\boxed{L = C\left((1/\omega C)^2 + R^2 \right)}$$

$$\tan\phi = \frac{1}{\omega RC}$$



Similarly...

$$\frac{1}{Z} = \frac{1}{Z_{RC}} \cos\phi$$

$$Z = Z_{RC}^2 / R$$

$$\boxed{Z = \frac{(1/\omega C)^2 + R^2}{R}}$$

→ The LRC combination is resistive (no reactance), so the current and voltage will be in phase