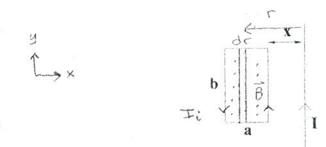


$B(r) = \frac{\mu_0 I}{2\pi r}$ (Ampere's Law)

$dA = b\,dr$ (out of page)

$d\Phi = B(r)\,dA$

1) In the diagram above, a rectangular conducting loop (dimensions a and b, resistance R) and a long straight wire that carries an electrical current I are both oriented so that they sit in the plane of the page. They will, for the duration of the problem, remain in the plane of the page with the wire carrying an electrical current I parallel to the right side of the conducting loop at a distance x to that side (as shown).

- 1a) (10 points) Assuming the conducting loop remains fixed in space while the wire is pulled away at a speed v_x, what is the magnitude of the resulting current induced in the loop? In what direction is that induced current traveling on the side closest to the wire (with or against the current direction in the wire?)

$d\vec{A}$ points out of the page, then:

$$\Phi_B = \int_x^{x+a} \frac{\mu_0 I}{2\pi r} b\,dr$$

$$\Phi_B = \frac{\mu_0 I b}{2\pi} \ln\left(\frac{x+a}{x}\right)$$

$$\mathcal{E}_i = -\frac{d\Phi_B}{dt}$$

$$\mathcal{E}_i = \frac{\mu_0 I b}{2\pi}\left(\frac{1}{x} - \frac{1}{x+a}\right)\frac{dx}{dt}$$

$$\mathcal{E}_i = \frac{\mu_0 I a b}{2\pi x (x+a)} V_x$$

$(\mathcal{E}_i > 0 \Rightarrow CCW)$

$I_i = \mathcal{E}_i / R$ (Still CCW)

$$\boxed{I_i = \frac{\mu_0 I a b V_x}{2\pi x (x+a) R}}$$

I_i is directed parallel to the Current in the long straight wire along the side of the loop closest to the wire

- 1b) (10 points) How large and in what direction is the force exerted on the loop by the wire?

By Symmetry $\vec{F}_{Top} + \vec{F}_{BOT} = 0$ So we need only Concern ourselves with the sides... Take $\hat{x}$ to point to the right, towards the wire...

$$\vec{F} = I\vec{L}\times\vec{B}$$

$$\vec{F}_L = I_i b \frac{\mu_0 I}{2\pi (x+a)} (-\hat{x})$$

$$\vec{F}_R = I_i b \frac{\mu_0 I}{2\pi x} (\hat{x})$$

$$\vec{F} = \frac{\mu_0 I b I_i a}{2\pi x (x+a)} \hat{x}$$

$\Rightarrow$ we have I_i in part a...

$$\boxed{F = \left[\frac{\mu_0 I a b}{2\pi x (x+a)}\right]^2 \frac{V_x}{R}}$$

directed towards the long Straight wire

- 1b) *Continued...*

- 1c) (5 points) How large and in what direction is the force exerted on the wire by the conducting loop? Is this result consistent with Lenz's Law? Explain.

Use Newton's 3rd law ‥ The force on the wire is

$$\boxed{F = \left[\frac{\mu_0 I a b}{2\pi x (x+a)}\right]^2 \frac{V_x}{R}}$$

directed towards the Conducting loop

← This is definitely Consistent with Lenz's Law – The induced current creates a force on the wire that would Counter the outward drift that Creates it...

- 1d) (5 points) Find the net torque on the conducting loop. For full credit, the grader must be able to follow the logic of your calculation.

$$\vec{\mu} = NI\vec{A}$$

$$\vec{\mu} = I_i ab\,\hat{z}$$

$$\vec{\mu} = \frac{\mu_0 I a^2 b^2 V_x}{2\pi x (x+a) R}\hat{z}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{k}$$

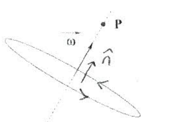
$$\vec{\tau} = \vec{\mu}\times\vec{B}$$

$\Rightarrow$ $\boxed{\tau = 0}$!!

$\vec{\mu}$ and $\vec{B}$ are already aligned

$$I = \frac{Q}{T} = \frac{Q\omega}{2\pi}$$

Uniform Ring
Q, R



2) A uniform circular ring of charge Q and radius R rotates around its longitudinal symmetry axis with an angular velocity $\vec{\omega}$, as shown.

- 2a) (5 points) Find the (vector) magnetic dipole moment of the ring.

$$\vec{\mu} = NIA\hat{n}$$

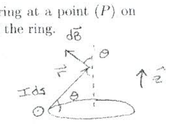
$$\vec{\mu} = \frac{Q\omega}{2\pi}\pi R^2 \hat{n}$$

$(\vec{\omega} = \omega\hat{n})$

$\Rightarrow$ $\boxed{\vec{\mu} = \frac{1}{2}QR^2\vec{\omega}}$

← Note that this works for any sign of Q... Cool!

- 2b) (10 points) Find the (vector) magnetic field due to the ring at a point (P) on the longitudinal symmetry axis, a distance z from the center of the ring.



Biot-Savart

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\hat{r}}{r^2}$$

by symmetry $\vec{B} = \vec{B}_z = \int dB_z$

$$dB = \frac{\mu_0 I}{4\pi}\frac{ds\,(1)\sin(90^\circ)}{(R^2+z^2)}$$

$$dB = \frac{\mu_0 I\,ds}{4\pi(R^2+z^2)}$$

$$dB_z = dB\cos\theta = dB\frac{R}{\sqrt{R^2+z^2}}$$

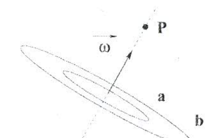
$$\int dB_z = \int \frac{\mu_0 I R\,ds}{4\pi(R^2+z^2)^{3/2}}$$

$$B_z = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$$

$I = \frac{Q\omega}{2\pi}$, and $\vec{\omega} = \omega\hat{z}$

$$\boxed{\vec{B} = \frac{\mu_0 Q R^2}{4\pi(R^2+z^2)^{3/2}}\vec{\omega}}$$

Note that $\vec{B} \propto \vec{\mu}$ ... Not TERRIBLY SURPRISING ‥



- 2c) (15 points) Now, let's replace the ring with a washer of inner-radius a and outer-radius b that carries a surface charge density

$$\sigma(r) = \frac{Q}{2\pi\ln(b/a)}\frac{1}{r^2}$$

where r measures the distance from the center of the washer to a point within the washer. Find the (vector) magnetic field due to the ring at a point (P) on the longitudinal symmetry axis, a distance z from the center of the ring.

$\Rightarrow$ Build the washer out of infinitesimal rings, each of which Contributes...

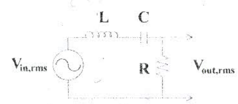
$$d\vec{B} = \frac{\mu_0\,dq\,r^2}{4\pi(r^2+z^2)^{3/2}}\vec{\omega}$$

$$dq = \sigma(r)\,dA = \frac{Q}{2\pi\ln(b/a)\,r^2}2\pi r\,dr = \frac{Q\,dr}{\ln(b/a)\,r}$$

$$d\vec{B} = \frac{\mu_0 Q\,2r\,dr}{8\pi\ln(b/a)(r^2+z^2)^{3/2}}\vec{\omega}$$

$$\vec{B} = \frac{\mu_0 Q\vec{\omega}}{8\pi\ln(b/a)}\int_a^b \frac{2r\,dr}{(r^2+z^2)^{3/2}}$$

$$\boxed{\vec{B} = \frac{\mu_0 Q\vec{\omega}}{4\pi\ln\left(\frac{b}{a}\right)}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]}$$



Filters are classified by the frequency-range of the signals they deliver to their output. High-pass and low-pass filters preferentially pass high- and low-frequency signals, respectively. Band-pass filters preferentially pass signals that have frequencies within some range (or 'band') of frequencies. Notch filters actually remove signals whose frequencies lie within some range of frequencies.

- 3a) (5 pts) Qualitatively discuss how the components in the circuit above will react to input signals over a very broad range of frequencies (say, 0 Hz to ∞ Hz). Explain how their respective behaviors, taken together, will make the circuit shown above behave like a band-pass filter.

The inductor presents small opposition to low frequencies and large opposition to high frequencies. The Capacitor presents large opposition to low frequencies and small opposition to high... In series, the combination will present large opposition to low and high frequencies and relatively modest opposition to intermediate frequencies. The largest current (and therefore greatest Vout) will occur for some 'band' of frequencies in the middle...

- 3b) (10 pts) If the signal on the input has an rms voltage $V_{in,rms}$ and a frequency ω, how large is the rms voltage at the output? At what frequency will this output voltage be greatest?

$$X_L = \omega L \quad X_C = \frac{1}{\omega C}$$

$$I_{rms} = \frac{V_{in,rms}}{Z}$$

$$V_{out,rms} = I_{rms} R$$

$$V_{out,rms} = V_{in,rms} \frac{R}{Z}$$

$$V_{out,rms} = \frac{V_{in,rms} R}{\sqrt{(X_L - X_C)^2 + R^2}}$$

$$\boxed{V_{out,rms} = \frac{V_{in,rms} R}{\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}}}$$

$V_{out,max}$ occurs when $\omega L = 1/\omega C$ $\Rightarrow \omega = \frac{1}{\sqrt{LC}}$

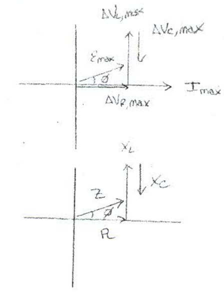
- 3c) (5 pts) By what phase angle will the output voltage lead or lag the input voltage? Under what conditions will it lead? ...lag?

Since Vout is in phase with I, this is equivalent to asking by how much will I lead or lag Vin...

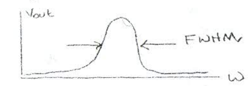
$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - 1/\omega C}{R}$$

Vout leads Vin if $1/\omega C > \omega L$

Vout lags Vin if $1/\omega C < \omega L$



- 3d) (10 pts) Do a quick, qualitative sketch of the output voltage vs. the frequency of the input signal. The width of that peak, $\Delta\omega$, is usually taken to be the "Full Width at Half-Maximum" (FWHM) - the distance (in frequency-space) between the two points at which the output voltage amplitude is half the input amplitude. Find the bandwidth of this filter, and discuss how one might achieve a sufficiently narrow peak without sacrificing output amplitude.



$$\frac{1}{2} = \frac{R}{\sqrt{(\omega L - 1/\omega C)^2 + R^2}}$$

$$4R^2 = (\omega L - 1/\omega C)^2 + R^2$$

$$\omega L - \frac{1}{\omega C} = \pm\sqrt{3}\, R$$

$$L\omega^2 \mp \sqrt{3}R\omega - \frac{1}{C} = 0$$

$$\omega = \frac{\pm\sqrt{3}R (\pm)\sqrt{3R^2 + 4L/C}}{2L}$$

The $\pm$'s are not correlated, so be careful...

$$\omega = \frac{\sqrt{3}R}{2L}\left[1 \pm \sqrt{1 + \frac{4}{3}\frac{L}{R^2C}}\right]$$

$$\omega = \frac{\sqrt{3}R}{2L}\left[-1 \pm \sqrt{1 + \frac{4}{3}\frac{L}{R^2C}}\right]$$

Pick the physically relevant solutions ($\omega > 0$)

$$\omega_{hi} = \frac{\sqrt{3}R}{2L}\left[1 + \sqrt{1 + \frac{4}{3}\frac{L}{R^2C}}\right]$$

$$\omega_{lo} = \frac{\sqrt{3}R}{2L}\left[-1 + \sqrt{1 + \frac{4}{3}\frac{L}{R^2C}}\right]$$

$$\boxed{\Delta\omega = \sqrt{3}\,\frac{R}{L}}$$

Minimizing R will minimize the width of the peak!