MT1 Physics 1C(2), F13

Full Name (Printed)

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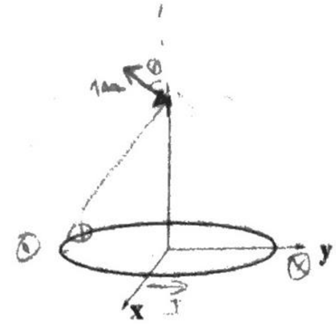
Student ID Number

Seat Number ______________________________

Problem	Grade
1	25 /30
2	20 /30
3	23 /30
Total	68 /90

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**



- 1a) (10 points) A circular, conducting loop of radius R lies in the x,y-plane, centered on the origin. A current I flows through the loop such that at $x = +R$, the current is headed in the $+\hat{y}$ direction, and at $x = -R$ the current is headed in the $-\hat{y}$ direction. Derive the resultant magnetic field (magnitude and direction) at every point along the z-axis.

By symmetry, all components of $\vec{B}$ are 0 except the $\hat{k}$ component.

$\vec{B} = B\hat{k}$

$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$

$dB = \frac{\mu_0}{4\pi} \frac{I dl}{r^2} \cos\theta$

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$B = \int dB = \int \frac{\mu_0}{4\pi} \frac{I dl}{r^2} \cos\theta = \frac{\mu_0}{4\pi} \frac{I}{r^2} \cos\theta (2\pi R)$ $\quad \cos\theta = \frac{R}{r}$

$\vec{B} = \frac{\mu_0 I R^2}{4\pi r^3} (2\pi)\hat{k} = \boxed{\frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \hat{k}}$

- 1b) (10 points) Let's replace the loop with an infinitesimally-thin, uniform ring of electric charge that extends from r to $r + dr$. If the surface charge-density on the ring is given by σ and the ring rotates about the z-axis with a constant angular velocity $\vec{\omega}$ (recall, the direction of $\hat{\omega}$ is obtained by using the right-hand rule with the physical motion of points on the ring) find the magnitude and direction of the (infinitesimal) magnetic field produced at every point along the z-axis.

$I = \frac{dq}{dt} \qquad \sigma = \frac{dq}{dA} \qquad \omega = \frac{d\theta}{dt}$

$dA = r\,dr\,d\theta \qquad I = \frac{\sigma\, dA}{dt} = \sigma r\, dr \frac{d\theta}{dt} = \sigma r\, dr\, \omega$

$d\vec{B} = \frac{\mu_0 I r^2}{2(r^2+z^2)^{3/2}} \hat{k} = \boxed{\frac{\mu_0 \sigma \omega\, dr\, r^2}{2(r^2+z^2)^{3/2}} \hat{k}}$

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- 1c) (10 points) Now lets replace the thin ring of part b with a washer that extends from $r = a$ to $r = b$. Charge is distributed over the washer with a surface charge density

$$\sigma(r) = \frac{q\,ab}{2\pi\,(b-a)}\,\frac{1}{r^3}$$

and it rotates with a constant angular velocity $\vec{\omega}$. Find the magnitude and direction of the magnetic field produced at every point on the z-axis.

$$\vec{B} = \int d\vec{B} = \int_a^b \frac{\mu_0\,\sigma(r)\,\omega\,r^2\,dr}{2\,(r^2+z^2)^{3/2}}\,\hat{k}$$

$$= \frac{\mu_0\,\omega}{2}\,\hat{k}\int_a^b \frac{\sigma(r)\,r^2}{(r^2+z^2)^{3/2}}\,dr$$

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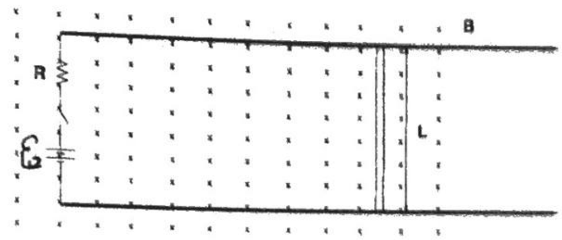
$$= \frac{\mu_0\,\omega\,q\,a\,b}{4\pi(b-a)}\int_a^b \frac{1}{(r^2+z^2)^{3/2}\,r}\,dr$$



2) A circuit is constructed with a battery ($\mathcal{E}$), a resistor (R), a switch, two long, parallel, horizontal conducting rails separated by a distance L and a conducting slider (that can move without friction over the rails) of mass m and length L. The whole apparatus is completely immersed in a strong, downward, uniform magnetic field $\vec{B}$. We'll assume, for the sake of simplicity, that this external field is so strong we can safely ignore any additional magnetic field contributed by the current through the rails (in reality, you probably don't want to do that).

- 2a) (5 points) What happens when the switch is closed? Describe in as much qualitative detail as you can, the directions of the current in the circuit, the force on the slider and the resulting motion of the rail.

A clockwise current goes through the circuit. The current goes from top to bottom through the bar. $\vec{F}_B = I d\vec{L} \times \vec{B}$, so by the right hand rule the magnetic force pushes the bar to the right. The rail will move to the right, eventually reaching some constant velocity. why?

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- 2b) (10 points) Find the magnitude and direction of the induced EMF in the circuit and the induced current at an instant when the slider is moving with a speed v_x. Clearly explain how those directions are obtained from the mathematical calculation and explain how they are consistent with Lenz's law.

$$\mathcal{E}_i = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} = \boxed{-BLv_x}$$

1 1

This is an $|\mathcal{E}_i| = BLv_x$ in the opposite direction of $\mathcal{E}$ from the battery. 1 1

Lenz's law says that an induced $\mathcal{E}_i$ must oppose the $\mathcal{E}$ from the battery. 2

Define signs Better

I from (1) + (2)

- 2c) (5 points) What is the magnitude and direction of the total current flowing in the circuit? What is the magnitude and the direction of the resulting force on the slider?

$$\mathcal{E}_{net} = IR$$

$$\mathcal{E} - |\mathcal{E}_i| = IR \qquad I = \frac{\mathcal{E} - BLv_x}{R}$$ 3 Direction?

$$\vec{F} = I\vec{L} \times \vec{B}$$

$$\vec{F} = ILB\hat{x} = \boxed{LB\left(\frac{\mathcal{E} - BLv_x}{R}\right)\hat{x}}$$ 1 1

$\hat{x}$ = to the right — good

- 2d) (10 points) Assuming the slider starts at rest the instant the switch is closed ($t = 0$), find the velocity of the slider at every instant after the switch is closed. Explain by first principles why you know the slider will reach a terminal velocity. What is the speed of the slider at terminal velocity?

$v(t)$?

2 $m\frac{dv}{dt} = \frac{LB\mathcal{E}}{R} - \frac{L^2B^2}{R}v_x \qquad v(0) = 0$

$\frac{dv}{dt} = -\frac{L^2B^2}{mR}v_x$

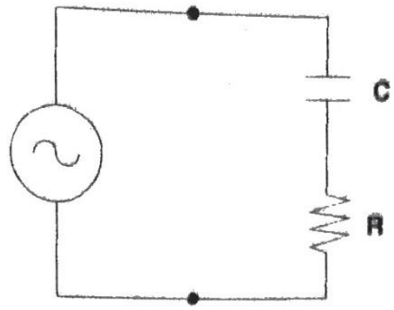
$\frac{dv}{v} = -\frac{L^2B^2}{mR}dt$

$e^{-\frac{L^2B^2}{mR}t} \longrightarrow 0$

If the slider never reached terminal velocity, it would get infinite energy where $E = \frac{1}{2}mv^2$

$V \longrightarrow V_{term} \neq \infty$

You loose this term ... it's important. It's the battery!



3) An capacitor (C) and a resistor (R) are connected in series across a source of alternating EMF ($\xi(t) = \xi_{max}\cos(\omega t)$). You may do the following calculations in complex space, but keep in mind, the final answers must all be real.

- 3a) (10 points) What is the impedance of RC combination? What is the amplitude of the current that passes through it? Does the current through this impedance lead or lag the voltage across it? By how much?

Z? I_{max}? Lead/lag?

$\tilde{Z} = \frac{1}{i\omega C} + R \qquad Z = \sqrt{R^2 + \left(-\frac{1}{\omega C}\right)^2} = \boxed{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = Z}$

$\hat{V} = \hat{I}\hat{Z} \qquad \hat{V} = \mathcal{E}_{max} e^{i\omega t}$

$\hat{Z} = R - i\frac{1}{\omega C} \qquad \theta = \tan^{-1}\left(\frac{-\frac{1}{\omega C}}{R}\right) \qquad \theta < 0$

$\tilde{Z} = Ze^{i\theta} \qquad \tilde{I} = \frac{\mathcal{E}_{max}}{Z} e^{i(\omega t - \theta)}$

so current leads voltage by $\tan^{-1}\left(\frac{1}{\omega C R}\right)$

$I(t) = Re[\hat{I}] = \frac{\mathcal{E}_{max}}{Z}\cos(\omega t - \theta)$

$\boxed{I_{max} = \frac{\mathcal{E}_{max}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}}$

- 3b) (10 points) What is the voltage amplitude across the resistor? What is the voltage amplitude across the capacitor? Will the sum of these voltage amplitudes equal the voltage amplitude of the source? Explain.

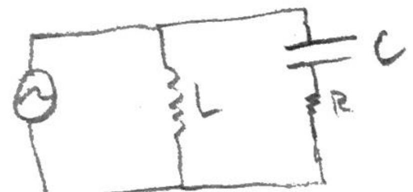
$V_{max_R} = I_{max_R} R = \boxed{\frac{\mathcal{E}_{max} R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} = V_{max_R}}$

$V_{max_C} = I_{max_C} Z_C = I_{max} X_C = \boxed{\frac{\mathcal{E}_{max}\left(\frac{1}{\omega C}\right)}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}} = V_{max_C}$

These will not sum to the voltage amplitude of the source because $V_C(t)$ is out of phase with $V_R(t)$

- 3c) (10 points) For a lot of good reasons, it is often desirable to present the source with a purely resistive load. We can tune-out the reactance in the RC network by adding an inductor in parallel with it (across the two dots in the circuit). What value should the inductor have? What will be the value of the new impedance seen by the source? Will current through this new LRC combination lead or lag the voltage across it? By how much?

L? Z? lead or lag?



$\mathcal{E}(t) = \mathcal{E}_{max}\cos(\omega t)$

$\tilde{V} = \tilde{I}_1 \tilde{Z}_1 \quad \hat{Z}_1 = iX_L = i\omega L$

$\tilde{V} = \tilde{I}_2 \tilde{Z}_2 \quad \hat{Z}_2 = R - i\frac{1}{\omega C}$

$1/\hat{Z}_{net} = 1/\hat{Z}_1 + 1/\hat{Z}_2 = \frac{1}{i\omega L} + \frac{1}{R - i\frac{1}{\omega C}}$ ✓

$1/\hat{Z}_{net} = \frac{i\omega L + R - i\frac{1}{\omega C}}{(i\omega L)(R - i\frac{1}{\omega C})} = \frac{R + i(\omega L - \frac{1}{\omega C})}{iR\omega L + L/C}$

$\frac{R + i(\omega L - 1/\omega C)}{(\frac{L}{C})^2 - (R\omega L)^2} \cdot (L/C - R\omega L i)$

purely resistive: $\mathrm{Im}(\hat{Z}) = 0$

$\frac{\frac{RL}{C} - R^2\omega L i + \frac{L}{C}(\omega L - \frac{1}{\omega C})i + R\omega L(\omega L - \frac{1}{\omega C})}{(\frac{L}{C})^2 - (R\omega L)^2}$

$Z_{net} = \frac{1}{(\frac{L}{C})^2 - (R\omega L)^2}\sqrt{\left(\frac{RL}{C} + R\omega L(\omega L - \frac{1}{\omega C})\right)^2 + \left(\frac{L}{C}(\omega L - \frac{1}{\omega C}) - R^2\omega L\right)^2}$

$\tilde{V} = \tilde{I}\tilde{Z}$

$I_{total} = \mathrm{Re}[\tilde{I}_{tot}] = \mathrm{Re}\left[\frac{\mathcal{E}_0 e^{i\omega t}}{Z}\right]$

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