Final Exam 1CF20-4

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Full Name (Signature) [signature]

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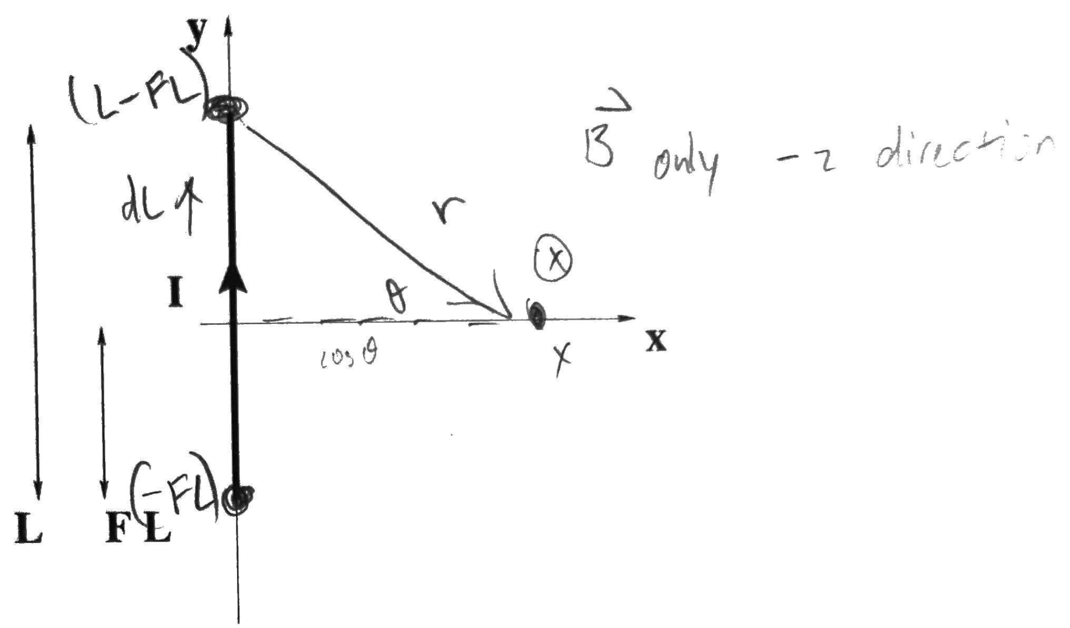
- The exam is open-book and open notes. You will probably do better to limit yourself to a single page of notes you prepared well in advance.
- **All work must be your own.** You are not allowed to collaborate with anyone else, you are not allowed to discuss the exam with anyone until all the exams have been submitted (after the close of the submissions window for the exam).
- You have **120 minutes** to complete the exam and more than sufficient time to scan the exam and upload it to GradeScope. The exam *must* be uploaded to GradeScope within the time alloted (that is, by the end of the 3-hour finals slot). We will only except submissions through GradeScope and will not accept any exam submitted after the submission window closes (CAE students must contact Corbin for instructions).
- **Given the limits of GradeScope, you must fit your work for each part into the space provided.** You may work on scratch paper, but you will not be able to upload the work you do on scratch paper, so it is essential that you copy your complete solution onto the exam form for final submission. We can only consider the work you submit on your exam form.
- **For full credit the grader must be able to follow your solution from first principles to your final answer. *There is a valid penalty for confusing the grader.***
- It is **YOUR** responsibility to make sure the exam is scanned correctly and uploaded before the end of the submission window. The graders may refuse to grade pages that are significantly blurred, solutions to problems that are not written in the correct place, pages submitted in landscape mode and/or work that is otherwise illegible - if any of this occurs, you may not receive *any* credit for the affected parts.
- Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**

The following must be signed before you submit your exam:

By my signature below, I hereby certify that all of the work on this exam was my own, that I did not collaborate with anyone else, nor did I discuss the exam with anyone while I was taking it.

Signature [signature]



1) A wire of finite length L carries a current I in the positive direction along the y-axis. A fraction F of that length is below the origin.

- **1a)** (10 points) Derive the vector magnetic field at all points on the $+x$-axis from appropriate fundamental equations assuming a right-handed coordinate system.

$d\vec{L}\times\hat{r}$ always same direction

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{L}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{I\,|d\vec{L}\times\hat{r}|\,(-\hat{z})}{r^2}$$

perpendicular component

$$d\vec{B} = -\hat{z}\,\frac{\mu_0 I}{4\pi r^2}\,dy\cos\theta$$

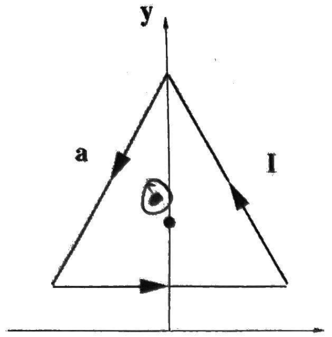
$$\left\{\begin{aligned} y &= x\tan\theta \\ dy &= \frac{x\,d\theta}{\cos^2\theta} \\ r &= \frac{x}{\cos\theta} \end{aligned}\right.$$

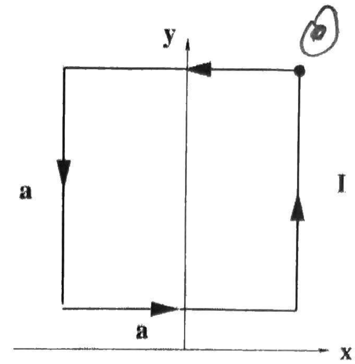
$$d\vec{B} = -\hat{z}\,\frac{\mu_0 I}{4\pi}\,\frac{\cancel{\cos^2\theta}}{x^2}\,\frac{x\,d\theta}{\cancel{\cos^2\theta}}\,\cos\theta$$

$$d\vec{B} = -\hat{z}\,\frac{\mu_0 I}{4\pi x}\int_{\theta_{-FL}}^{\theta_{L-FL}}\cos\theta\,d\theta$$

$$\vec{B} = -\hat{z}\,\frac{\mu_0 I}{4\pi x}\,\sin\theta\,\Big|_{\theta_{-FL}}^{\theta_{L-FL}}$$

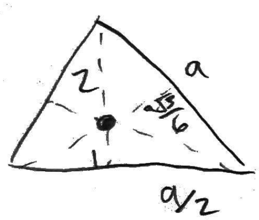
$$\vec{B} = -\hat{z}\,\frac{\mu_0 I}{4\pi x}\left[\frac{L-FL}{\sqrt{(L-FL)^2+x^2}} - \frac{(-FL)}{\sqrt{(FL)^2+x^2}}\right]$$



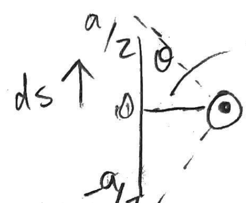


- **1b) (15 points)** Find the magnitude and direction of the magnetic field at the center of the equilateral triangle of side-length a shown above if it carries a current I.

Find $\vec{B}$ due to One Segment, Because point is at center, total $\vec{B} = 3\vec{B}_{\text{segment 1}}$



$dB = \frac{\mu_0 I}{4\pi r^2} ds \times \hat{r}$



$R = \frac{a\sqrt{3}}{6}$

$\sin\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2+(s)^2}}$

$h^2 + \frac{a^2}{4} = a^2$

$h^2 = \frac{3a^2}{4}$

$h = \frac{a\sqrt{3}}{2}$

$\frac{1}{3}h = \frac{a\sqrt{3}}{6}$

$\frac{1}{2}B_1 = \int_0^{a/2} \frac{\mu_0 I}{4\pi r^2} ds \sin\theta$

$B_1 = \frac{\mu_0 I}{2\pi}\int_0^{a/2} \frac{R}{((s)^2+R^2)^{3/2}} ds$

$B_1 = \frac{\mu_0 I}{2\pi}\left[\frac{(s)}{R\sqrt{s^2+R^2}}\right]_0^{a/2}$

$s = r\tan u$

$u = \tan^{-1}\left(\frac{s}{r}\right)$ $ds = r\sec^2 u\, du$

$\frac{R}{[s^2+R^2]^{3/2}} ds$

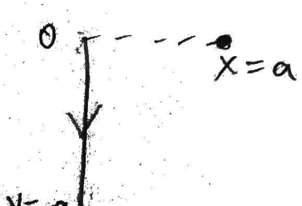
$\int \frac{r\sec^2 u}{(r^2\tan^2 u + r^2)^{3/2}} du \quad \frac{1}{r^2}\int \frac{1}{\sec u} du = \frac{\sin u}{r^2} = \frac{s}{r^3\sqrt{\frac{s^2}{r^2}+1}}$

$\boxed{B_{tot} = \frac{9\mu_0 I}{\pi a\sqrt{3}}\left[\frac{a/2}{\sqrt{\frac{1}{4}a^2 + \frac{1}{12}a^2}}\right]}$

- **1c) (5 points)** Find the magnitude and direction of the magnetic field on the top-right corner of the square of side-length a shown above if it carries a current I.

only bottom and left Sides produce a B field
– same B field

$\vec{B} = +\hat{z}\frac{\mu_0 I}{4\pi x}\left[\frac{b}{\sqrt{b^2+x^2}} - \frac{a}{\sqrt{a^2+x^2}}\right]$



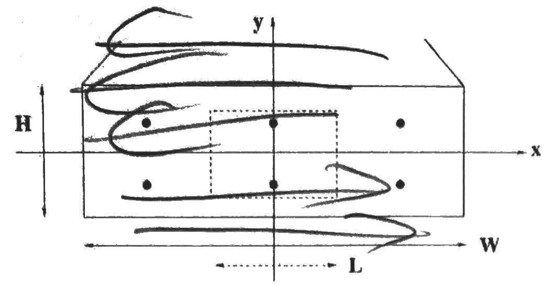
$\vec{B} = +\hat{z}\frac{\mu_0 I}{4\pi a}\left[0 + \frac{a}{\sqrt{a^2+a^2}}\right]$

$\vec{B} = \hat{z}\frac{\mu_0 I}{4\pi a}\left[\frac{a}{\sqrt{2a^2}}\right] = \frac{\mu_0 I}{4\pi a}\left(\frac{1}{\sqrt{2}}\right)$

$b = 0$

$a = -a$

$\boxed{\vec{B}_{tot} = 2\vec{B} = \frac{\mu_0 I}{2\pi a}\left(\frac{1}{\sqrt{2}}\right)\hat{z}}$



2) A wide current sheet of dimension $W \times H$ (where $W \gg H$), centered on the origin, carries a total electric current I out of the page. The current density associated with this current looks like:

$$\vec{J} \propto \left(\frac{2y}{H} - 1\right)^2 \hat{z}$$

w is large so it looks the same

- **2a) (10 points)** Normalize the current density.

$A = HW$

$dA = wdy$

$I_{enc} = \int J \, dA$

$I = \int J dA = \int c\left(\frac{2y}{H} - 1\right)^2 w dy$

$I = cw \int \left(\frac{2y}{H} - 1\right)^2 dy$ $\quad u = \frac{2y}{H} - 1$

$I = cw \int u^2 \frac{H du}{2}$ $\quad \frac{du}{dy} = \frac{2}{H}$

$I = \frac{cw\left(\frac{2y}{H} - 1\right)^3 H}{6}$

$c = \frac{6I}{w\left(\frac{2y}{H} - 1\right)^3 H}$

- **2b) (10 points)** Find the amount of current enclosed in a rectangular loop of width L that extends vertically from $-y$ to $+y$. Consider all values of y.

Rectangular loop $\quad y < H$

$I_{enc} = \int J \cdot dA = \int_{-y}^{y} \frac{6I}{w\left(\frac{2y}{H} - 1\right)^3 H} w dy$ $\quad u^{-3}$ $\quad \frac{1}{u^3} \frac{H du}{2}$

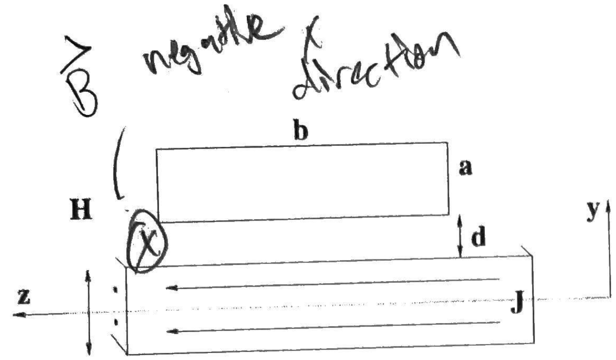
Widths cancel out here??

$I_{enc} = \frac{6I}{H} \int_{-y}^{y} \frac{1}{\left(\frac{2y}{H} - 1\right)^3} dy$ $\quad u = \frac{2y}{H} - 1$ $\quad \frac{du}{dy} = \frac{2}{H}$

$I_{enc} = \frac{6I}{H}\left[-\frac{h}{4\left(\frac{2y}{H} - 1\right)^2}\right]_{-y}^{y} = \boxed{\frac{6I}{H}\left(-\frac{2h^4 y}{16y^4 - 8h^2y^2 + h^4}\right)}$

$y > H$

$I_{enc} = I \frac{L}{W}$



Side View of Current Sheet

- **2c) (5 points)** (Disregard the diagram above for a moment.) Find the vector magnetic field at all points on the $+y$-axis.

$y < H$

$$2B(y)x = \mu_0 I_{enc}$$

$$2B(y)x = \mu_0 \cdot \frac{6I}{H}\left(\frac{-2h^4 y}{16y^4 - 8h^2y^2 + h^4}\right)$$

$$B(y) = (\hat{-x})\frac{\mu_0 3I}{Hx}\left(\frac{-2h^4 y}{16y^4 - 8h^2y^2 + h^4}\right)$$

$y > H$

$$2B(y)x = \mu_0 I \frac{L}{W}$$

$$B = \frac{\mu_0 I L}{2xW} = \boxed{\frac{\mu_0 I}{2W}}$$

- **2d) (5 points)** A rectangular conducting loop with N turns is positioned above and parallel to the current sheet, in the y, z-plane, as shown. It has a height a, a width b and the closest side is a distance d from the sheet. How large is the mutual inductance between the loop and the sheet?

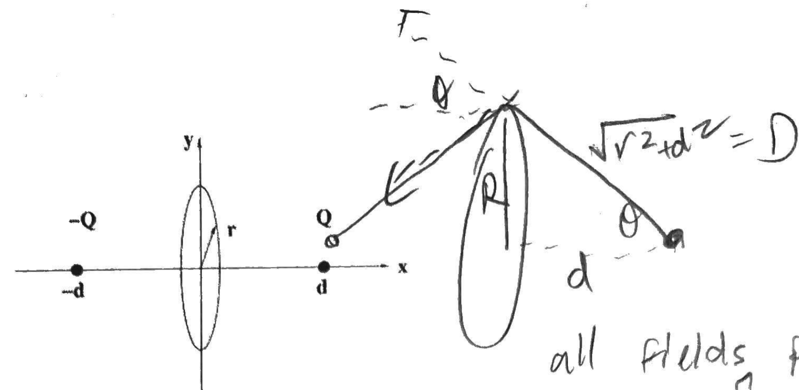
$$\text{Mutual Inductance} = \left|\frac{\mathcal{E}_1, N_2^{(2)}}{dI_1/dt}\right|$$

$y > H$ $\quad \Phi_B = \frac{\mu_0 I}{2W} ab$

$$\mathcal{E}_1 = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 ab}{2\pi}\frac{dI}{dt}$$

$$\boxed{\mathcal{E}_1 = -N \cdot \frac{\mu_0 ab}{2\pi}\frac{dI}{dt}}$$

Inductance due to changing current of sheet



all fields point in $-\hat{x}$

3) Electric charges Q and $-Q$ are placed on the x-axis at $x = d$ and $x = -d$, respectively.

- **3a) (10 points)** Find the vector electric field for points in the y, z-plane a distance r from the origin.

$$E = 2k\frac{Q}{D^2} = (-\hat{x})\frac{2kQ}{(r^2+d^2)}\cos\theta \qquad \cos\theta = \frac{d}{\sqrt{r^2+d^2)}}$$

$$E = (-\hat{x})\frac{2kQd}{(r^2+d^2)^{3/2}}$$

- **3b) (10 points)** Find the flux of the electric field through a circular disk of radius r that is located in the y, z-plane, centered on the origin.

$$E = \frac{2kQd}{(r^2+d^2)^{3/2}} \qquad s = 2\pi r \quad ds = 2\pi dr$$

Flux at one ring / E points along dA already

$$dF = Eds = \frac{2kQd}{(r^2+d^2)^{3/2}} 2\pi dr$$

$$dF = \frac{4\pi kQd}{(r^2+d^2)^{3/2}} dr$$

$$\boxed{\Phi = \int d\Phi = \int_0^R \frac{4\pi kQd}{(r^2+d^2)^{3/2}} dr}$$

- **3c) (5 points)** Let's assume that positive charge is being transferred from $x = -d$ to $x = +d$ at a rate I. How large is the displacement current that flows through a disk of radius r that is located in the y, z-plane, centered on the origin?

$$I_{\text{Displacement}} = \varepsilon_0 \frac{d\Phi_E}{dt}$$

E changes

area stays the same

$$I_{\text{displacement}} = \varepsilon_0 (E \cdot dA) \frac{d}{dt}$$

$$\frac{dq}{dt} = I$$

$$I_{\text{displacement}} = \varepsilon_0 \pi r^2 \left(\frac{2k\, Id}{(r^2+d^2)^{3/2}} \right)$$

- **3d) (5 points)** Evaluate *Ampere's* law using that disk of radius r and discuss what you find.

Amperes law

$$\oint B \cdot ds = \mu_0 \overset{0}{\cancel{I}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint B \cdot ds = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$B(\cancel{2\pi r}) = \mu_0 \varepsilon_0 \pi r^{\cancel{2}} \left(\frac{\cancel{2}kId}{(r^2+d^2)^{3/2}} \right)$$

$$B = \frac{\mu_0 \varepsilon_0 \pi r\, kId}{(r^2+d^2)^{3/2}}\, \hat{\phi}$$

Then is a B field in the azmuthal direction

4) Let's design an instrument to observe the radiation emitted by a cosmic object some very large distance R away at a wavelength λ.

- 4a) (5 points) Suppose the object radiates energy isotropically at rate P_0. If we use a dish antenna that has a diameter D and assume a lossless path, at what rate will we recover energy from the object in our receiver?

$$P_{object} = \left(\frac{P_0}{4\pi R^2}\right)\pi\left(\frac{D}{2}\right)^2 = \boxed{\frac{P_0 D^2}{16 R^2}}$$

- 4b) (5 points) How large (in distance) is the smallest feature we can resolve using this single dish?

circular

$$\theta_{min} = \frac{1.22\lambda}{D}$$

object size

$$\frac{S}{R} = \frac{1.22\lambda}{D} \qquad \boxed{S = \frac{1.22\lambda R}{D}}$$

- 4c) (10 points) Now let's consider using, instead, two dishes, each of diameter $\frac{D}{2}$ separated laterally by a distance $2D$. At what rate will we recover energy from the object? What is the smallest feature (in distance) that we can resolve?

$$P_{object\ total} = 2P_{object} = 2\left(\frac{P_0}{4\pi R^2}\right)\pi\left(\frac{D}{4}\right)^2$$

$$= 2\left(\frac{P_0 D^2}{64 R^2}\right) = \frac{P_0 D^2}{32 R^2}$$

Smallest feature

$$\theta_{min} = \frac{1.22\lambda}{(D/2)}$$

$$\frac{S}{R} = \frac{1.22\lambda}{D/2} \qquad \boxed{S = \frac{2.44\lambda R}{D}}$$

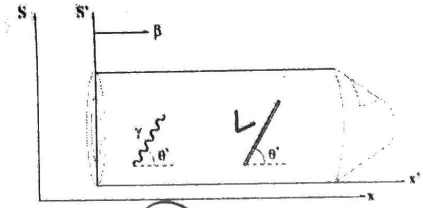
- **4d) (5 points)** Discuss the relative merits of using a single dish vs. two dishes for our observation.

Using 2 dishes of lesser diameter allows for the system to capture smaller features, however, the rate at which it absorbs energy is reduced as well.

The single dish system has the opposite tradeoff. It recovers energy quicker but can't see as small ~~of~~ an object like the 2 dish system.

- **4e) (5 points)** If you ever find yourself out near Socorro, NM, I would *strongly* recommend taking a side-trip to the "VLA", where 27 25m dishes are used in combination for observations. Is ganging so many antennas together mostly about signal strength or resolution? Discuss.

Using so many dishes in combination, the power absorbed would be much less than a single disk with a greater diameter. Ganging this many antennas together is mostly an attempt at enhancing signal resolution, possibly due to the easier assembly overall as well. The focusing power / ability to see smaller objects is much greater



$\left(\frac{v}{c}\right)^2$ $\left(\frac{Bc}{c}\right)^2$ B^2

5) A rocket moves through the lab with a speed βC. A stick is inclined in the rocket at an angle θ' as shown. A passenger in the rocket fires a photon parallel to the stick.

- 5a) (5 points) What angle does the stick make with the x-axis in the lab?

$\Delta x' = L_0 \cos\theta_0$ $\quad \Delta x = \frac{\Delta x'}{\gamma}$ $\quad \Delta y = \Delta y'$

$\Delta y' = L_0 \sin\theta_0$

$\tan\theta^0 = \frac{\Delta y''}{\Delta x'}$ $\quad \tan\theta_{lab} = \frac{\Delta y}{\Delta x}$

$\tan\theta_{lab} = \frac{1}{\sqrt{1-B^2}} \frac{\Delta y'}{\Delta x'}$

$\tan\theta_{lab} = \frac{1}{\sqrt{1-B^2}} \tan\theta^0$

$\theta_{lab} = \tan^{-1}\left(\frac{1}{\sqrt{1-B^2}} \tan\theta^0\right)$

- 5b) (5 points) What angle does the photon make with the x-axis in the lab?

$\theta'_{lab} = \tan^{-1}\left(\frac{1}{\sqrt{1-B^2}} \tan\theta^0\right)$

$B \rightarrow$ different than B of stick

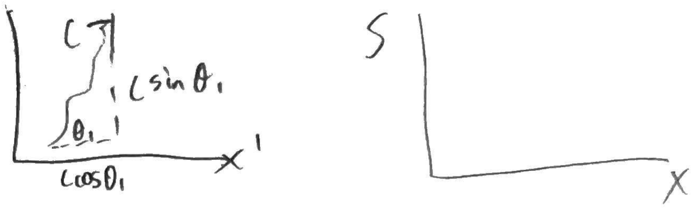
- 5c) (5 points) Compare your first two answers and discuss.

Angle of the stick and the photon are different. The photon is moving with an x and y velocity with respect to the rocket while the stick is at rest. The dilation constant gamma will be larger

Find velocity in lab

• 5d) (10 points) Show that the photon is moving through the lab with the expected speed.

Photon has to still be moving at speed c



X direction $\dfrac{u_{\parallel}}{c} = \dfrac{\frac{c\cos\theta_1}{c} + \beta}{1 + \beta\frac{c\cos\theta_1}{c}} = \dfrac{\cos\theta_1 + \beta}{1 + \beta\cos\theta_1}$

y direction $u_{\perp} = \dfrac{\frac{c\sin\theta_1}{c}}{\gamma\left(1 + \beta\frac{c\cos\theta_1}{c}\right)} = \dfrac{\sin\theta_1}{\gamma(1 + \beta\cos\theta_1)} = \dfrac{\sin\theta_1\sqrt{1-\beta^2}}{(1+\beta\cos\theta_1)}$

$c^2 = x^2 + y^2$

$1 = \left(\dfrac{\cos\theta_1 + \beta}{1 + \beta\cos\theta_1}\right)^2 + \left(\dfrac{\sin\theta_1\sqrt{1-\beta^2}}{1 + \beta\cos\theta_1}\right)^2$?? true

• 5e) (5 points) Pick two points along the photon's trajectory in the rocket. When the photon hits the first point we'll call that **Event 1** and when the photon hits the second point, we'll call that **Event 2**. If the elapsed time between those events is $\Delta t'$ in the rocket, what is the elapsed time between the events as seen in the lab? Briefly discuss.

$\Delta t = \gamma \Delta t'$

$\Delta t = \dfrac{1}{\sqrt{1 - 1^2}} \Delta t'$

The elapsed time seen in the lab is nearing infinity. This shows that there is infinite time dilation for photons and no time passes in the frame of the poton,