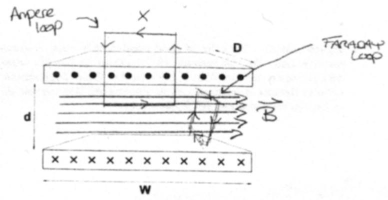
Ampere's Law:

$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$

$Bx = \mu_0 K x$

$B = \mu_0 K$



1) Two thin, parallel, conducting sheets of dimension $D \times W$ are separated by a distance d as shown ($D >> W >> d$). The sheets each carry a *linear* current density K, one into the plane of the page, one out, as shown.

- 1a) (15 points) Use Faraday's law to calculate the self-inductance of the arrangement.

The Faraday loops will actually run perpendicular to the lines of $\vec{B}$. Take $d\vec{A}$ parallel to $\vec{B}$

$\Phi_B = \int \vec{B}\cdot d\vec{A}$

$\Phi_B = \mu_0 K D d$

$K = I/W$

$\Phi_B = \frac{\mu_0 I D d}{W}$

$\mathcal{E}_i = -\frac{d\Phi_B}{dt}$

$\mathcal{E}_i = -\frac{\mu_0 d D}{W}\frac{dI}{dt}$

$L = \left|\frac{\mathcal{E}_i}{\frac{dI}{dt}}\right| \Longrightarrow \boxed{L = \frac{\mu_0 d D}{W}}$

- 1b) (10 pts) Verify your answer to the first part using energy considerations.

$u_B = \frac{1}{2\mu_0}B^2$

$u_B = \frac{1}{2}\frac{\mu_0}{W^2}I^2$

$U = \int u_B\, dV$

$U = \frac{1}{2}\frac{\mu_0}{W^2}I^2\, dDW$

$U = \frac{1}{2}\frac{\mu_0 dD}{W}I^2 \longleftrightarrow$ Compare to $U = \frac{1}{2}LI^2$

$\boxed{L = \frac{\mu_0 d D}{W}}$

- 1c) (5 pts) Define $\hat{L}$ as the inductance per unit length (measured along the current) and $\hat{C}$ as the capacitance per unit length. Calculate $\frac{1}{\sqrt{\hat{L}\hat{C}}}$. This quantity plays an important role in the practical evaluation of transmission lines. Care to guess what it is?

$\hat{L} = \frac{\mu_0 d}{W}$ $\quad \hat{C} = \frac{\epsilon_0 W}{d}$

$\boxed{\frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{\mu_0\epsilon_0}}}$ ← Essentially a fundamental Constant.

... With dimensionality $\frac{L}{T}$

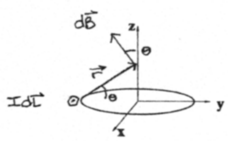
... (the speed of light in Vacuum)

Symmetry:

$\vec{B} = B_z\hat{z}$

$B_z = \int dB_z$

$dB_z = dB\cos\theta$



$\cos\theta = \frac{R}{\sqrt{R^2+z^2}}$

- 2a) (10 points) A circular, conducting loop of radius R lies in the x,y-plane, centered on the origin. A current I flows through the loop such that at $x = +R$, the current is headed in the $+\hat{y}$ direction, and at $x = -R$ the current is headed in the $-\hat{y}$ direction. Derive the resultant magnetic field (magnitude and direction) at every point along the z-axis.

$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{L}\times\hat{r}}{r^2}$

$dB = \frac{\mu_0 I}{4\pi}\frac{|d\vec{L}\times\hat{r}|}{r^2}$

$dB = \frac{\mu_0 I}{4\pi}\frac{dL}{(R^2+z^2)}$

$dB_z = dB\cos\theta$

$dB_z = \frac{\mu_0 I R\, dL}{4\pi(R^2+z^2)^{3/2}}$

$B_z = \frac{\mu_0 I R}{4\pi(R^2+z^2)^{3/2}}\int dL$

$\boxed{\vec{B} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\hat{z}}$

- 2b) (10 points) Let's replace the loop with an infinitesimally-thin, uniform ring of electric charge that extends from r to $r + dr$. If the surface charge-density on the ring is given by σ and the ring rotates about the z-axis with a constant angular velocity $\vec{\omega}$ (recall, the direction of $\hat{\omega}$ is obtained by using the right-hand rule with the physical motion of points on the ring) find the magnitude and direction of the (infinitesimal) magnetic field produced at every point along the z-axis.

Assume $\hat{\omega} = \hat{z} \Rightarrow$ then our ring is well-described by the work we did in part a

$d\vec{B} = \frac{\mu_0\, dI\, r^2}{2(r^2+z^2)^{3/2}}\hat{\omega} \longleftarrow dI = \frac{dq}{T} = \frac{\sigma dA}{2\pi/\omega} = \sigma\omega r\, dr$

$d\vec{B} = \frac{\mu_0\sigma\omega r^3 dr}{2(r^2+z^2)^{3/2}}\hat{\omega}$

$\boxed{d\vec{B} = \frac{\mu_0\vec{\omega}}{2}\cdot\frac{\sigma r^3 dr}{(r^2+z^2)^{3/2}}}$

- 2c) (10 points) Now lets replace the thin ring of part b with a washer that extends from $r = a$ to $r = b$. Charge is distributed over the washer with a surface charge density

$$\sigma(r) = \frac{q\,ab}{2\pi\,(b-a)}\,\frac{1}{r^3}$$

and it rotates with a constant angular velocity $\vec{\omega}$. Find the magnitude and direction of the magnetic field produced at every point on the z-axis.

Superpose an infinite number of infinitesimal rings "

$d\vec{B} = \frac{\mu_0\vec{\omega}}{2}\frac{qab}{2\pi(b-a)}\frac{dr}{(r^2+z^2)^{3/2}}$

$d\vec{B} = \frac{\mu_0\vec{\omega}qab}{4\pi(b-a)}\frac{z\,d\theta}{\cos^2\theta}\frac{\cos^3\theta}{z^3}$

$\int d\vec{B} = \frac{\mu_0\vec{\omega}qab}{4\pi(b-a)z^2}\int d\theta\cos\theta$

$\vec{B} = \frac{\mu_0 qab\vec{\omega}}{4\pi(b-a)z^2}\sin\theta\Big|_{\theta_a}^{\theta_b}$

$r = z\tan\theta$

$dr = \frac{z\,d\theta}{\cos^2\theta}$

$\sqrt{r^2+z^2} = \frac{z}{\cos\theta}$

$\boxed{\vec{B} = \frac{\mu_0 qab\vec{\omega}}{4\pi(b-a)z^2}\left[\frac{b}{\sqrt{b^2+z^2}} - \frac{a}{\sqrt{a^2+z^2}}\right]}$