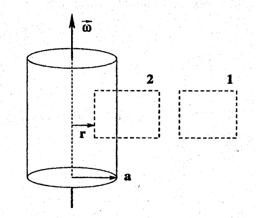


1) A *very long*, solid, uniform cylinder of radius a and uniform electric charge density ρ rotates around its longitudinal symmetry axis with an angular velocity $\vec{\omega}$

- **1a)** (5 points) Using symmetry arguments and first principles, predict the **direction** of the magnetic field the cylinder produces at points inside and outside the cylinder.

There are no magnetic monopoles, so the lines of magnetic field must form closed loops around the current. The cylinder is long and cylindrically symmetric, so $\vec{B}$ cannot depend on longitudinal or azimuthal coordinates... These two facts, taken together, rule out radial and azimuthal components to $\vec{B}$, so $\vec{B} = B(r)\hat{z}$ (where $\hat{z}$ points in the direction of $\vec{\omega}$). Since $\vec{B}$ must loop around I in the right-hand sense...

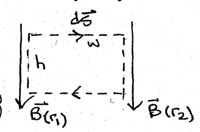
(if $\rho>0$) $\vec{B}$ is parallel to $\vec{\omega}$ $(r<a)$	(if $\rho<0$) $\vec{B}$ is opposite $\vec{\omega}$ $(r<a)$
$\vec{B}$ is opposite $\vec{\omega}$ $(r>a)$	$\vec{B}$ is parallel to $\vec{\omega}$ $(r>a)$

- **1b)** (5 points) Using the loop labeled '1', discuss the rate at which the magnetic field outside the cylinder changes with respect to changes in radial distance from the symmetry axis of the cylinder. Make an argument for the value of the magnetic field at large distances from the symmetry axis and deduce the value of the magnetic field at points outside the cylinder.

Ampere's Law: $\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$

$$(B(r_2) - B(r_1))h = 0$$

$$B(r_1) = B(r_2)$$ ← $\vec{B}$ is uniform outside the cylinder



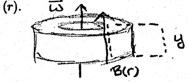
Far from the cylinders, we'd expect $B(r \gg a) \to 0$

So $\boxed{B(r) = 0 \text{ outside the cylinder}}$

- **1c)** (5 points) Now, if you stop to think about it, there's a significant chance that your answer to part b is in conflict with your answer to part a. Explain. Do your best to resolve this discrepancy.

The cylinder can't be infinitely long, and therefore the fields are likely to bulge a bit (that is, to have radial components that depend on longitudinal position)... This will lead to small path-integral contributions along the top and bottom of loop 1 that allow $B(r_1)$ to differ slightly from $B(r_2)$

- **1d)** (10 points) Find the rate at which **charge** passes through the loop marked '2' as a function of the distance from the inner boundary of the loop to the symmetry axis (r).



$$I_{enc} = \frac{q}{T} = \frac{\rho\pi(a^2-r^2)y\omega}{2\pi}$$

$$\boxed{I_{enc} = \frac{1}{2}\rho(a^2-r^2)y\omega}$$

- **1e)** (5 points) Find the magnetic field at all values of the radial distance from the symmetry axis.

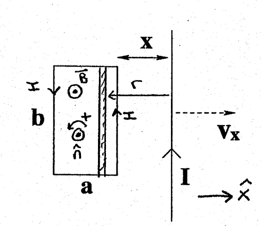
$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$$

$$B(r)\,y = \mu_0 \tfrac{1}{2}\rho(a^2-r^2)y\omega$$

$\vec{B} \uparrow\uparrow \vec{\omega}$ if $\rho>0$

$$\boxed{\begin{array}{ll} \vec{B}(r) = \frac{1}{2}\mu_0\rho(a^2-r^2)\vec{\omega} & (r<a) \\ \vec{B}(r) = 0 & (r>a) \end{array}}$$

$d\Phi_B = B\,dA$
$= Bb\,dr$
$= \frac{\mu_0 I b}{2\pi}\frac{dr}{r}$



Ampere's Law

$$B = \frac{\mu_0 I}{2\pi r}$$

2) In the diagram above, a rectangular conducting loop (dimensions a and b, resistance R) and a long straight wire that carries an electrical current I are both oriented so that they sit in the plane of the page. They will, for the duration of the problem, remain in the plane of the page with the wire carrying an electrical current I parallel to the right side of the conducting loop at a distance x to that side (as shown).

- **2a)** (10 points) Assuming the conducting loop remains fixed in space while the wire is pulled away at a speed v_x, what is the magnitude of the resulting current induced in the loop? In what direction is that induced current traveling on the side closest to the wire (with or against the current direction in the wire?)

$$\Phi_B = \int d\Phi_B = \frac{\mu_0 I b}{2\pi}\int_x^{x+a}\frac{dr}{r}$$

$$\Phi_B = \frac{\mu_0 I b}{2\pi}\ln\left(\frac{x+a}{x}\right)$$

$$\mathcal{E}_i = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 I b}{2\pi}\left(\frac{1}{x+a} - \frac{1}{x}\right)\frac{dx}{dt}$$

$$\mathcal{E}_i = \frac{\mu_0 I b}{2\pi}\left(\frac{1}{x} - \frac{1}{x+a}\right)V_x$$

← $V_x > 0$, so $\mathcal{E}_i > 0$ it runs in + azimuthal direction (CCW)

$$I_i = \mathcal{E}_i/R$$

So does I_i :)

$$\boxed{I_i = \frac{\mu_0 I b}{2\pi R}\left(\frac{1}{x} - \frac{1}{x+a}\right)V_x}$$

On the side closest to the wire, I_i runs in the same direction as the current in the wire

- **2b)** (10 points) How large and in what direction is the force exerted on the loop by the wire? How large and in what direction is the force exerted on the wire by the conducting loop? Is this result consistent with Lenz's Law? Explain.

$\vec{F}_B = I\vec{L}\times\vec{B}$. Symmetry & right-hand rule result in the forces on the top and bottom of the loop canceling. We need only worry about the sides. Newton's 3rd law says $\vec{F}$ on wire equal and opposite force on loop...

on loop:

$$\vec{F}(x+a) = I_i bB(-\hat{x}) = \frac{\mu_0 I b^2}{2\pi R}\left(\frac{1}{x} - \frac{1}{x+a}\right)\frac{\mu_0 I}{2\pi(x+a)}(-\hat{x}) = \frac{\mu_0^2 I^2 b^2}{4\pi^2 R}\left(\frac{1}{x} - \frac{1}{x+a}\right)\frac{1}{x+a}(-\hat{x})$$

$$\vec{F}(x) = I_i bB(\hat{x}) = \frac{\mu_0 I b^2}{2\pi R}\left(\frac{1}{x} - \frac{1}{x+a}\right)\frac{\mu_0 I}{2\pi x}(\hat{x}) = \frac{\mu_0^2 I^2 b^2}{4\pi^2 R}\left(\frac{1}{x} - \frac{1}{x+a}\right)\frac{1}{x}(\hat{x})$$

The net force on the loop: $\vec{F}(x+a) + \vec{F}(x) = \frac{\mu_0^2 I^2 b^2}{4\pi^2 R}\left(\frac{1}{x} - \frac{1}{x+a}\right)^2\hat{x}$

The net force on the wire (N3) $= -\frac{\mu_0^2 I^2 b^2}{4\pi^2 R}\left(\frac{1}{x} - \frac{1}{x+a}\right)^2\hat{x}$

If $\hat{x}$ points to right:

$$\boxed{\begin{array}{l} \vec{F}_{on\ loop} = \frac{1}{R}\left[\frac{\mu_0 I b}{2\pi}\left(\frac{1}{x} - \frac{1}{x+a}\right)\right]^2\hat{x} \\ \vec{F}_{on\ wire} = -\frac{1}{R}\left[\frac{\mu_0 I b}{2\pi}\left(\frac{1}{x} - \frac{1}{x+a}\right)\right]^2\hat{x} \end{array}}$$

It is the motion of the wire away from the loop that creates the induced current in the loop. That induced current pulls the wire back towards the loop – Consistent with Lenz's law!

- **2c)** (10 points) Find the net torque on the conducting loop. For full credit, the grader must be able to follow the logic of your calculation.

At first blush, this problem isn't helped by the fact that $\vec{B}$ varies across the loop. How do you evaluate $\vec{\tau} = \vec{\mu}\times\vec{B}$? But then the realization hits that everywhere $\vec{\mu}$ is parallel to $\vec{B}$... the dipole moment is already aligned with the field and the net torque is zero!

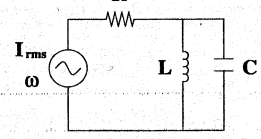
Put another way (as a check), if we use the wire as the location of the rotational axis, the lines of relevant force pass through the rotation axis ⇒ net torque is zero.

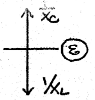
$$\boxed{\vec{\tau} = 0}$$

$X_L = \omega L$
$X_C = 1/\omega C$

$\Delta V_{R,RMS} = I_{RMS} R$
$\Delta V_{LC,RMS} = I_{RMS} Z_{LC}$

parallel:





$\frac{1}{Z_{LC}} = \frac{1}{X_C} - \frac{1}{X_L}$
$\frac{1}{Z_{LC}} = \omega C - \frac{1}{\omega L}$
$Z_{LC} = \frac{1}{\omega C - 1/\omega L}$

3) An RLC network is driven by a sinusoidally-varying electric current of root-mean-square value I_{rms} and angular frequency ω.

- 3a) (5 points) Find the root-mean-square values of the voltage that appears across the resistor and the voltage that appears across the LC-network.

$\Delta V_{R,RMS} = I_{RMS} R$

$\Delta V_{LC,RMS} = I_{RMS} \frac{1}{\omega C - 1/\omega L}$

Interesting question: What happens at resonance and why? ::

- 3b) (5 points) Will the sum of the rms-voltages across the resistor and the LC-network add up to the rms-voltage across the current source? Why or why not? Explain.

No Because there is a phase difference between the voltages that the simple sum ignores...

- 3c) (5 points) Under what conditions will the voltage across the LC network lead the driving current? Under what conditions will it lag? By how much will it lead or lag in each case?

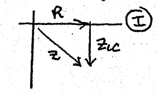
If $1/X_C$ is larger than $1/X_L$, Current will lead Voltage...
So... If X_C is larger than X_L, Voltage will lead Current...

If $\omega < \frac{1}{\sqrt{LC}}$ the voltage across the LC network will lead the current through it by 90°

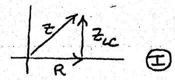
If $\omega > \frac{1}{\sqrt{LC}}$ the voltage will lag the current by 90°

- 3d) (10 points) What is the impedance seen by the source?

(If $\omega < \frac{1}{\sqrt{LC}}$)



(If $\omega > \frac{1}{\sqrt{LC}}$)



(Regardless of ω...) $Z = \sqrt{R^2 + Z_{LC}^2}$

$Z = \sqrt{R^2 + \left(\frac{1}{\omega C - 1/\omega L}\right)^2}$

- 3e) (5 points) Find the root-mean-square value of the voltage across the current source. How does your answer compare to your response in part b?

$\mathcal{E}_{rms} = I_{rms} Z$

$\mathcal{E}_{rms} = I_{rms} \sqrt{R^2 + \left(\frac{1}{\omega C - 1/\omega L}\right)^2}$

Definitely Not the Simple Sum of $\mathcal{E}_{R,RMS}$ and $\mathcal{E}_{LC,RMS}$ ::