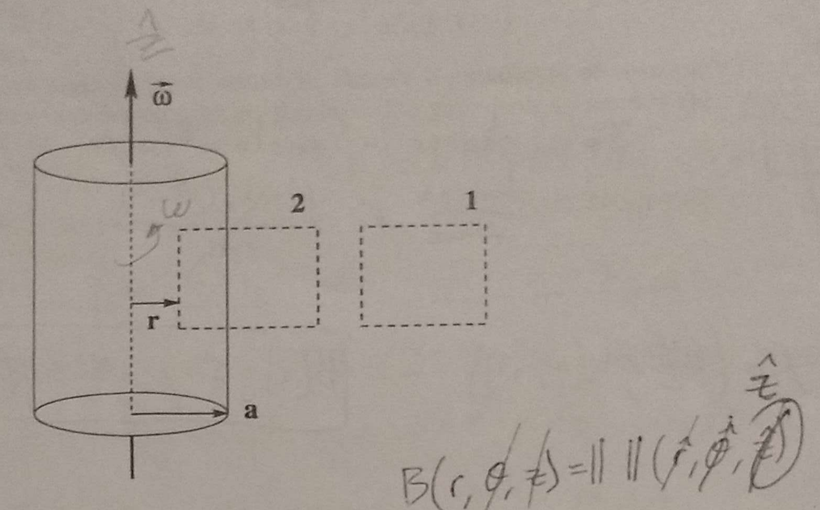
Seat Number ______________________

Problem	Grade
1	24 /30
2	16 /30
3	17 /30
Total	57 /90

57

- Do not peek at the exam until you are told to begin. You will have approximately 50 minutes to complete the exam.
- Don't spend too much time on any one problem. Solve 'easy' problems first. Go for partial credit!
- **HINT:** Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**



1) A *very long*, solid, uniform cylinder of radius a and uniform electric charge density ρ rotates around its longitudinal symmetry axis with an angular velocity $\vec{\omega}$

- 1a) (5 points) Using symmetry arguments and first principles, predict the **direction** of the magnetic field the cylinder produces at points inside and outside the cylinder.

~~The mag~~ $\vec{B}$ will point up at points inside the cylinder and point down outside the cyl. Following one volume of charge around the axis it creates a toroidal mag. field. As all points in the cylinder create identical loops, the variances of $d\vec{B}$ along the x-y axes cancel out leaving just the $\hat{z}$ component.

- 1b) (5 points) Using the loop labeled '1', discuss the rate at which the magnetic field outside the cylinder changes with respect to changes in radial distance from the symmetry axis of the cylinder. Make an argument for the value of the magnetic field at large distances from the symmetry axis and deduce the value of the magnetic field at points outside the cylinder.

$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$ $I_{enc} = 0$

Far away from the symmetry axis the $\vec{B} = 0$ as there is no charge enclosed by the loop.

Outside the cylinder this will hold true, so $\vec{B}(r) = 0$ for $r > a$.

4

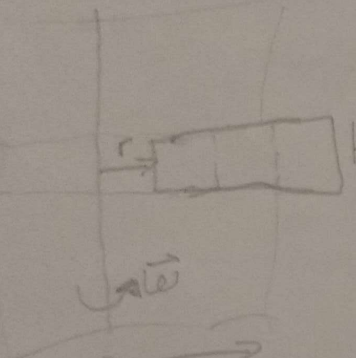
- 1c) (5 points) Now, if you stop to think about it, there's a significant chance that your answer to part b is in conflict with your answer to part a. Explain. Do your best to resolve this discrepancy.

Though the $+\hat{z}$ components of $\vec{B}$ are contained within the finite cross-section of the inside of the cylinder, the $-\hat{z}$ components of $\vec{B}$ to satisfy the ~~m~~ dipole requirement of a magnetic field is dispersed throughout an infinite area.
Identical ~~to~~ to a long solenoid, the mag. of $\vec{B}$ thus drops to 0 for each finite point outside the cylinder.

10

- 1d) (10 points) Find the rate at which **charge** passes through the loop marked '2' as a function of the distance from the inner boundary of the loop to the symmetry axis (r). I

h-tall slice of cyl.



Charge in ~~rotated cross-section~~ ~~volume of cylinder~~

$= \rho \cdot \pi a^2 h$

Charge that passes through loop 2
(so charge w/ radius ~~...~~ $r \le x \le a$)

$= \rho h \cdot (\pi a^2 - \pi r^2)$

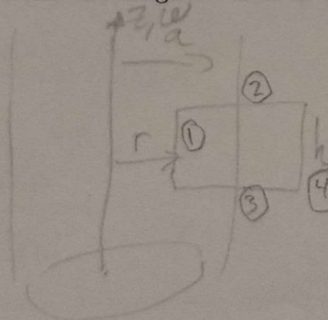
This charge passes through loop 2 once every $\omega/2\pi$ seconds so

$$I = \frac{\omega \rho h (a^2 - r^2)}{2}$$

- 1e) (5 points) Find the magnetic field at all values of the radial distance from the symmetry axis.

5

loop 2



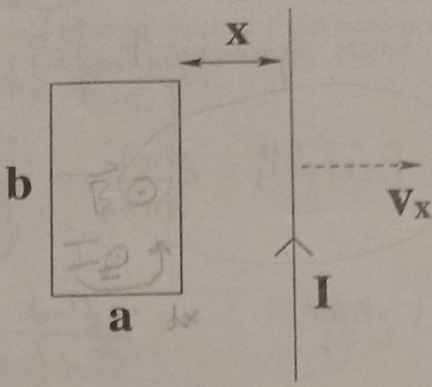
$I_{enc} = \frac{\omega \rho h}{2}(a^2 - r^2)$

$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$

$B \cdot ds_{2,3} = 0$

so $\oint \vec{B} \cdot d\vec{s} = Bh = \mu_0 I_{enc}$ $\quad B = 0$ for ds_4 $(r > a)$

$$\vec{B} = \mu_0 \frac{\omega \rho}{2}(a^2 - r^2)\,\hat{z}$$



2) In the diagram above, a rectangular conducting loop (dimensions a and b, resistance R) and a long straight wire that carries an electrical current I are both oriented so that they sit in the plane of the page. They will, for the duration of the problem, remain in the plane of the page with the wire carrying an electrical current I parallel to the right side of the conducting loop at a distance x to that side (as shown).

10 • 2a) (10 points) Assuming the conducting loop remains fixed in space while the wire is pulled away at a speed v_x, what is the magnitude of the resulting current induced in the loop? In what direction is that induced current traveling on the side closest to the wire (with or against the current direction in the wire?)

Amp loop

$\vec{B}\cdot 2\pi r = \mu_0 I$

$\vec{B} = \frac{\mu_0 I}{2\pi r}$ ✓

Flux through loop

$\Phi_B = \oint \vec{B}\cdot d\vec{A}$

$= \int_x^{x+a} (b\,dr)\cdot\left(\frac{\mu_0 I}{2\pi r}\right)$

$= \frac{b\mu_0 I}{2\pi}\ln\left(\frac{x+a}{x}\right)$ ✓

$\mathcal{E}_i = \frac{d\Phi_B}{dt}$

$= -\frac{b\mu_0 I}{2\pi}\cdot\frac{x}{x+a}\cdot\left[\frac{d}{dt}\left(\frac{x+a}{x}\right) = \frac{-1}{x^2}\cdot\frac{d}{dt}x\right]$

$= \frac{ab\mu_0 I}{2\pi}\cdot\frac{x}{x+a}\cdot\frac{1}{x^2}\cdot V_x$

$I_2 = \frac{\mathcal{E}}{R} = \frac{ab\mu_0 I}{2\pi R}\left(\frac{1}{x+a}\right)\left(\frac{V_x}{x}\right)$ ✓

Φ_B is decreasing so I_2 tries to increase field by axially traveling ⊙ out of the page. Thus the current on the right side of the loop is travelling with I. ✓

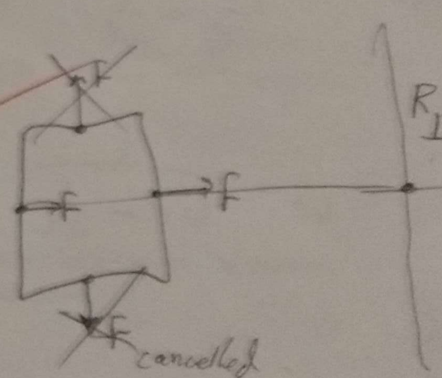
3 • 2b) (10 points) How large and in what direction is the force exerted on the loop by the wire? How large and in what direction is the force exerted on the wire by the conducting loop? Is this result consistent with Lenz's Law? Explain.

-1

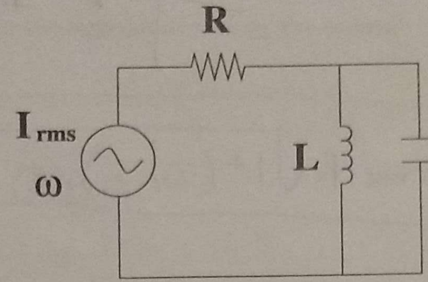
3 • 2c) (10 points) Find the net torque on the conducting loop. For full credit, the grader must be able to follow the logic of your calculation.

$\tau = F \times r = F \cdot x$

-1



$\tau_{net} = 0$



3) An RLC network is driven by a sinusoidally-varying electric current of root-mean-square value I_{rms} and angular frequency ω.

3

- 3a) (5 points) Find the root-mean-square values of the voltage that appears across the resistor and the voltage that appears across the LC-network.

$V = IR$

$V_{rms,R} = I_{rms} \cdot R$ ✓

$\frac{1}{X_{L,C}} = \frac{1}{\omega L} + \omega C = \frac{1 + \omega^2 LC}{\omega L}$

$X_{L,C} = \frac{\omega L}{1 + \omega^2 LC}$

$V_{rms,LC} = I_{rms} \cdot X_{LC}$

$= I_{rms} \cdot \frac{\omega L}{(1 + \omega^2 LC)}$

from 3d:

$X_{LC} = i\left(\frac{1 - \omega^2 LC}{\omega L}\right)$

$|X_{LC}| = \frac{1 - \omega^2 LC}{\omega L}$

$V_{rms,LC} = I_{rms} \cdot \frac{(1 - \omega^2 LC)}{\omega L}$ ✗

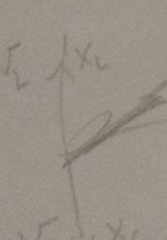
5

- 3b) (5 points) Will the sum of the rms-voltages across the resistor and the LC-network add up to the rms-voltage across the current source? Why or why not? Explain.

While the resistor is in phase with I_{rms}, the inductor (L) and cap. (C) may not be so they could share V_{rms} at different time ~~stamps~~ frames.

2

- 3c) (5 points) Under what conditions will the voltage across the LC network lead the driving current? Under what conditions will it lag? By how much will it lead or lag in each case?



$X_{LC} = i(X_L - X_C)$

V_{LC} will lead if $X_L > X_C$: $\omega L > \frac{1}{\omega C}$, $\omega^2 L > \frac{1}{C}$, $\omega^2 > \frac{1}{LC}$, $\omega > \sqrt{\frac{1}{LC}}$

V_{LC} will lag if: $\omega L < \frac{1}{\omega C}$, $\omega < \frac{1}{\sqrt{LC}}$

It will lead/lag by 90° as there is no real impedence component of X_{LC} and thus it is confined to i-axis.

← This seems odd.

5

- **3d) (10 points)** What is the impedance seen by the source?

$$\frac{1}{X_{LC}} = \frac{1}{X_C} + \frac{1}{X_L} = i\left(\omega C - \frac{1}{\omega L}\right)$$

$$X_{LC} = i\left(\frac{1}{\omega L} - \omega C\right) = i\left(\frac{1-\omega^2 CL}{\omega L}\right)$$

$$X_{LCR} = R + X_{LC} = R + i\left(\frac{1-\omega^2 CL}{\omega L}\right)$$ X

2

- **3e) (5 points)** Find the root-mean-square value of of the voltage across the current source. How does your answer compare to your response in part *b*?

$V_{rms} = I_{rms}|X|$

$$V_{rms} = I_{rms} \cdot \sqrt{R^2 + \left(\frac{1-\omega^2 LC}{\omega L}\right)^2}$$ X

The answer matches w/ part b as it takes the phase-desync of R and LC into account for the computation of |X|.