$\vec{\mu} = NI\vec{A}$

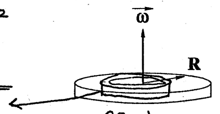
for a spinning ring: $Q, \vec{\omega}, R$

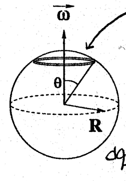
$\vec{\mu} = \frac{Q\vec{\omega}}{2\pi}\pi R^2$

$\vec{\mu} = \frac{1}{2}QR^2\vec{\omega}$

$dq = \sigma dA$

$dq = \frac{Q}{\pi R^2}2\pi r\,dr \rightarrow dq = \frac{2Q r\,dr}{R^2}$





$dq = \rho\, dV$

$dq = \frac{3Q}{4\pi R^3}\pi r^2 dz$

$r = R\sin\theta$

$z = R\cos\theta$

$dz = R\,d(\cos\theta)$

$dq = \frac{3Q}{4}\sin^2\theta\; d(\cos\theta)$

$dq = \frac{3Q}{4}(1-\cos^2\theta)\,d(\cos\theta)$

- **1a) (10 points)** A uniform **disk** of charge Q and radius R spins about its perpendicular symmetry axis with an angular velocity $\vec{\omega}$ (as shown in the left diagram above). Derive the (vector) magnetic dipole moment of the distribution.

Build the disk up with infinitesimal rings... $(dq, \vec{\omega}, r)$

$d\vec{\mu} = \frac{1}{2}dq\, r^2\vec{\omega}$

$d\vec{\mu} = \frac{1}{2}\frac{2Q}{R^2}r^3 dr\,\vec{\omega}$

$\vec{\mu} = \int d\vec{\mu} = \frac{Q\vec{\omega}}{R^2}\int_0^R dr\, r^3$

$\boxed{\vec{\mu} = \frac{1}{4}QR^2\vec{\omega}}$ ← direction checks out for $\pm Q$ ✓✓

- **1b) (10 points)** A uniform **sphere** of charge Q and radius R spins about an axis that passes through its center with an angular velocity $\vec{\omega}$ (as shown in the right diagram above). How large a contribution to the total magnetic dipole moment of the sphere will the infinitesimally-thin disk shown in the right diagram make?

for an infinitesimal disk $(dq, r, \vec{\omega})$...

From top diagram, $r = R\sin\theta$

$d\vec{\mu} = \frac{1}{4}dq\, r^2\vec{\omega}$

$d\vec{\mu} = \frac{1}{4}\frac{3Q}{4}(1-\cos^2\theta)\,d(\cos\theta)\; R^2(1-\cos^2\theta)\,\vec{\omega}$

$\boxed{d\vec{\mu} = \frac{3}{16}QR^2\vec{\omega}\,(1-\cos^2\theta)^2\,d(\cos\theta)}$

- **1c) (5 points)** Derive the (vector) magnetic dipole moment of the spinning spherical distribution.

$x \equiv \cos\theta$

$dx = d(\cos\theta)$

$\vec{\mu} = \int d\vec{\mu} = \frac{3}{16}QR^2\vec{\omega}\int_{-1}^{1}(1-x^2)^2\,dx$

$\vec{\mu} = \frac{3}{8}QR^2\vec{\omega}\int_0^1(1-2x^2+x^4)\,dx$

$\boxed{\vec{\mu} = \frac{1}{5}QR^2\vec{\omega}}$

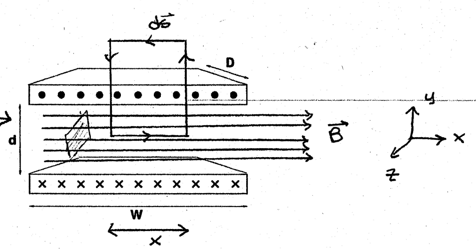
- **1d) (5 points)** When placed in a uniform magnetic field B, the sphere precesses with an angular frequency Ω. Assuming its mass is also uniformly distributed, find the charge-to-mass ratio for the sphere.

Recall from lecture, $\Omega = \gamma B_z$ where $\gamma \equiv \frac{\vec{\mu}}{\vec{L}} = \frac{\frac{1}{5}QR^2\vec{\omega}}{\frac{2}{5}mR^2\vec{\omega}} = \frac{1}{2}\frac{Q}{m}$

$\Omega = \frac{1}{2}\frac{Q}{m}B$

$\boxed{\frac{Q}{m} = \frac{2\Omega}{B}}$

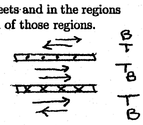
(b) →



2) Two thin, parallel, conducting sheets of dimension $D \times W$ are separated by a distance d as shown ($D \gg W \gg d$). The sheets each carry a *linear* current density K, one into the plane of the page, one out, as shown.

- **2a) (10 points)** Use your knowledge of the magnetic field in and around long, wide current sheets to obtain a qualitative description of the magnetic field in the region between the sheets and in the regions outside the sheets. Follow that up by actually finding the magnetic field in each of those regions.

Outside a long, wide sheet of current, the magnetic field is uniform, oriented along the sheet, directed by the right hand rule with the current. The contributions from the top (T) & bottom (B) sheet are equal in magnitude & directed as shown... The net field outside the sheets is zero ⁚)



referring to the amperian loop up top,

$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$

$Bx = \mu_0 K x$

$B = \mu_0 K$

$\boxed{\begin{array}{ll}\vec{B} = \mu_0 K\,\hat{x} & \text{between plates}\\ \vec{B} = 0 & \text{outside plates}\end{array}}$

- **2b) (10 points)** Use Faraday's law to calculate the self-inductance of the arrangement.

$\mathcal{E}_i = -\frac{d\Phi_B}{dt}$ ↗ where the relevant area is perpendicular to B and spans the height & length of the arrangement,

$\mathcal{E}_i = -\frac{d}{dt}(BDd)$

$\mathcal{E}_i = -\frac{d}{dt}(\mu_0 K D d)$

$\mathcal{E}_i = -\frac{d}{dt}(\mu_0 \frac{I}{W} D d)$

$\mathcal{E}_i = -\mu_0\frac{Dd}{W}\frac{dI}{dt}$

$L = \left|\frac{\mathcal{E}_i}{\frac{dI}{dt}}\right|$

$\boxed{L = \frac{\mu_0 D d}{W}}$

- **2c) (10 points)** Verify your answer for the self-inductance of the arrangement by using energy considerations.

$u_B = \frac{1}{2\mu_0}B^2$

$u_B = \frac{1}{2}\mu_0 K^2$

$u_B = \frac{1}{2}\mu_0\frac{I^2}{W^2}$

u_B is uniform!

$U_B = \int u_B\, dV$

$U_B = u_B\int dV$

$U_B = u_B(\text{Vol})$

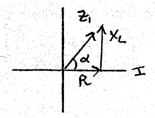
$U_B = \frac{1}{2}\mu_0\frac{I^2}{W^2}\,dDW$

but $U_B = \frac{1}{2}LI^2$ (since we picked up all the energy)

So...

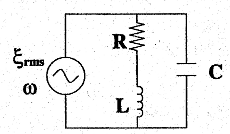
$\boxed{L = \frac{\mu_0 D d}{W}}$

①



$Z_1 = \sqrt{R^2 + X_L^2}$

① ②



$X_L = \omega L$
$X_C = 1/\omega C$

$Z_2 = X_C$

3) The following questions all refer to the circuit shown above. For convenience, we'll refer to the branch that contains the resistor and inductor as branch 1 and the branch that contains the capacitor as branch 2.

- 3a) (5 points) Find the RMS current through branch 1 and through branch 2.

$I_{rms} = \dfrac{\mathcal{E}_{rms}}{Z}$

$$\boxed{I_{rms,1} = \frac{\mathcal{E}_{rms}}{\sqrt{R^2 + (\omega L)^2}} \qquad I_{rms,2} = \mathcal{E}_{rms}\,\omega C}$$

Grader: for full credit, all derived answers must be in terms of R, ω, L, C → No X's or Z's

- 3b) (5 points) Will the current through branch 1 lead or lag the voltage across it? By how much?

With current as the reference for the series phase diagram, it looks like

Current lags Voltage by α, where $\tan\alpha = \dfrac{\omega L}{R}$

Consistent with 'ELI' ☺

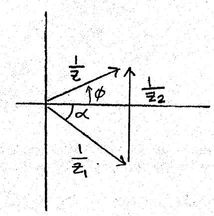
- 3c) (5 points) Will the current through branch 2 lead or lag the voltage across it? By how much?

Branch 2 only has the Capacitor...

Current leads Voltage by 90°

'ICE'

- 3d) (5 points) Sketch the phasor diagram that will describe how the impedance in branch 2 combines with the impedance in branch 1 to give us the effective impedance as seen by the voltage source. Clearly label the diagram with the relevant branch impedances (or inverse-impedances) and phase angles.



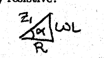
$\dfrac{1}{Z_1} = \dfrac{1}{\sqrt{R^2 + (\omega L)^2}}$

$\dfrac{1}{Z_2} = \omega C$

$\tan\alpha = \dfrac{\omega L}{R}$

- 3e) (5 points) At what frequency will the effective impedance seen by the source look purely resistive?

From the diagram: $\dfrac{1}{Z_2} = \dfrac{1}{Z_1}\sin\alpha$



$\sin\alpha = \dfrac{\omega L}{Z_1}$

$\dfrac{1}{Z_2} = \dfrac{\omega L}{Z_1^2}$

$\omega C = \dfrac{\omega L}{R^2 + (\omega L)^2}$

$$\boxed{\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}}$$

- 3f) (5 points) How large is the effective impedance when the driving frequency is tuned so that it is purely resistive?

$\dfrac{1}{Z} = \dfrac{1}{Z_1}\cos\alpha$

$\dfrac{1}{Z} = \dfrac{R}{Z_1^2}$

$(\omega L)^2 = \dfrac{L}{C} - R^2$

$R^2 + (\omega L)^2 = \dfrac{L}{C}$

$Z_1^2 = L/C$

$$\boxed{Z = \frac{RC}{L}}$$