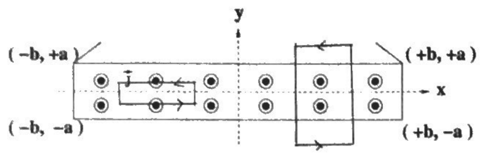


$2x \otimes 2y$

1) The picture above shows a long, wide conducting sheet of height $2a$, width $2b$ through which a current of uniform area-density $\vec{J} = \vec{J}_0[1-(y/a)^2]$ flows towards the observer.

- 1a) (15 points) Imagine a rectangular cross-section of width $2x$ and height $2y$, centered on the origin and oriented perpendicular to the flow of electric current. How much current flows through that cross-section if i) $x < b$ and $y < a$? ii) $x < b$ and $y > a$?

i) $I = \int \vec{J}\cdot d\vec{A} = \int_{-y}^{y} J_0\left[1-\frac{y^2}{a^2}\right]2x\,dy$

$I = 2xJ_0\left[y - \frac{1}{3}\frac{y^3}{a^2}\right]_{-y}^{y}$

$I = 4xyJ_0\left(1-\frac{1}{3}\frac{y^2}{a^2}\right)$

ii) $I = \int \vec{J}\cdot d\vec{A} = \int_{-a}^{a} J_0\left(1-\frac{y^2}{a^2}\right)2x\,dy$ ← the current cuts off at $y = \pm a$

$I = 4ax\,J_0\left(\frac{2}{3}\right)$

$I = \frac{8}{3}ax\,J_0$

$$\boxed{\begin{array}{ll} I = 4xy\left(1-\frac{1}{3}\frac{y^2}{a^2}\right)J_0 & (y<a) \\ I = \frac{8}{3}xa\,J_0 & (y>a) \end{array}}$$

- 1b) (5 points) In order to find the magnetic field due to this distribution of current, you will have to exploit a symmetry that is present in the problem. Clearly identify that symmetry and tell the grader how you will use it to calculate the magnetic field.

Symmetry: $\vec{B}(y) = -\vec{B}(-y)$ (that is, $B(y) = B(-y)$)
Place the top and bottom of our amperian loops at y and $-y$, respectively, so the magnitude of B is identical along both the relevant pieces of the loop...

- 1c) (10 points) Find the magnitude and direction of the magnetic field at any point along the y-axis.

$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$

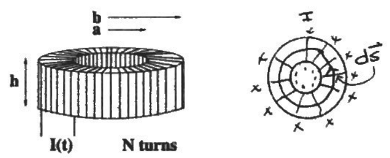
$B(y)\,2x + 0 + B(-y)\,2x + 0 = \mu_0 I_{enc}$

$B(y)\cdot 4x = \mu_0 I_{enc}$

$B(y) = \frac{\mu_0}{4x} I_{enc}$ ← use part a for I_{enc}

directed left when $y>0$, right when $y<0$

$$\boxed{\begin{array}{ll} (y<a) & B(y) = \mu_0 y\left(1-\frac{1}{3}\frac{y^2}{a^2}\right)J_0\,(-\hat{x}) \\ (y>a) & B(y) = \frac{2}{3}\mu_0 a J_0\left(-\frac{y}{|y|}\hat{x}\right) \end{array}}$$



2) Consider the toroidal solenoid of rectangular cross-section shown above. The height h, the radii a and b, and the time-varying current $I(t)$ are all known.

- 2a) (5 points) Find the magnitude of the magnetic field within the solenoid, as a function of radial distance from the center of the solenoid.

$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$

$B\,2\pi r = \mu_0 N I$

$$\boxed{B = \frac{\mu_0 N I}{2\pi r}}$$

- 2b) (10 points) Find the magnitude of the electromotive force induced across the leads of the solenoid.

$\Phi_{B,1} = \int \vec{B}\cdot dA = \int_a^b \frac{\mu_0 N I}{2\pi r} h\,dr$

$\Phi_{B,1} = \frac{\mu_0 N I h}{2\pi}\ln\left(\frac{b}{a}\right)$

$\mathcal{E}_{i,1} = -\frac{d\Phi_{B1}}{dt} = -\frac{\mu_0 N h}{2\pi}\ln\left(\frac{b}{a}\right)\frac{dI}{dt}$

$\mathcal{E}_{i,N} = N\mathcal{E}_{i,1}$

$$\boxed{|\mathcal{E}_{i,N}| = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)\left|\frac{dI}{dt}\right|}$$

- 2c) (5 points) What is the inductance of the solenoid?

$L = \left|\frac{\mathcal{E}_{i,N}}{\frac{dI}{dt}}\right|$

$$\boxed{L = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)}$$

- 2d) (5 points) Write (or derive) an the expression for the inductance of an N-turn cylindrical solenoid of radius a and longitudinal length x (where $x >> a$).

This is one of those things you should probably remember, but no shame if you can derive it quickly :)

$L = \mu_0 n^2 \pi a^2 x$

$$\boxed{L = \frac{\mu_0 N^2 \pi a^2}{x}}$$

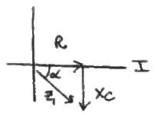
- 2e) (5 points) Show that in the limit $b - a = d << a$, the inductance of the toroidal solenoid reduces to the expression for the inductance of a long cylindrical solenoid of cross-sectional area hd and length $2\pi a$.

$\ln\left(\frac{b}{a}\right) = \ln\left(1+\frac{d}{a}\right) \approx \frac{d}{a}$ if $d << a$

$$\boxed{L \simeq \frac{\mu_0 N^2 hd}{2\pi a}}$$

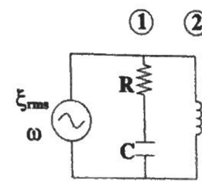
→ take $hd \to \pi a^2$
$2\pi a \to x$
⇒ get the long solenoid of part d :)

✓



$z_1 = \sqrt{R^2 + X_C^2}$

$\tan\alpha = \frac{X_C}{R}$



$Z_2 = X_L$

$X_C = 1/\omega C$

$X_L = \omega L$

3) The following questions all refer to the circuit shown above. For convenience, we'll refer to the branch that contains the resistor and capacitor as branch 1 and the branch that contains the inductor as branch 2.

- 3a) (5 points) Find the RMS current through branch 1 and through branch 2.

$I_{rms} = \frac{\mathcal{E}_{rms}}{Z}$

$$\boxed{I_{rms,1} = \frac{\mathcal{E}_{rms}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \qquad I_{rms,2} = \frac{\mathcal{E}_{rms}}{\omega L}}$$

Grader: For full credit all derived answers should be in terms of R, ω, C, L (no X's or Z's)

- 3b) (5 points) Will the current through branch 1 lead or lag the voltage across it? By how much?

For Series Circuits, we plot Voltage vs Current. From the plot...

Current leads Voltage by α, where $\tan\alpha = \frac{1}{R\omega C}$

Consistent with 'ICE'

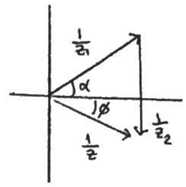
- 3c) (5 points) Will the current through branch 2 lead or lag the voltage across it? By how much?

Branch 2 is just an inductor...

Current lags Voltage by 90°

'ELI'

- 3d) (5 points) Sketch the phasor diagram that will describe how the impedance in branch 2 combines with the impedance in branch 1 to give us the effective impedance as seen by the voltage source. Clearly label the diagram with the relevant branch impedances (or inverse-impedances) and phase angles.

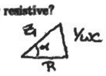


$\frac{1}{z_1} = \frac{1}{\sqrt{R^2 + (1/\omega C)^2}}$

$\frac{1}{z_2} = \frac{1}{\omega L}$

$\tan\alpha = \frac{1}{RC\omega}$

- 3e) (5 points) At what frequency will the effective impedance seen by the source look purely resistive?



$\frac{1}{z_2} = \frac{1}{z_1}\sin\alpha$

$\frac{1}{z_2} = \frac{1}{\omega C z_1^2}$

$\omega L = \omega C\left(R^2 + (1/\omega C)^2\right)$

$$\boxed{\omega = \frac{1}{\sqrt{LC - (RC)^2}}}$$

- 3f) (5 points) How large is the effective impedance when the driving frequency is tuned so that it is purely resistive?

$\frac{1}{z} = \frac{1}{z_1}\cos\alpha = \frac{R}{z_1^2}$

$z = \frac{1}{R} z_1^2$

$\left(\frac{1}{\omega C}\right)^2 = \frac{L}{C} - R^2$

$R^2 + \left(\frac{1}{\omega C}\right)^2 = L/C$

$z_1^2 = L/C$

$$\boxed{z = \frac{L}{RC}}$$