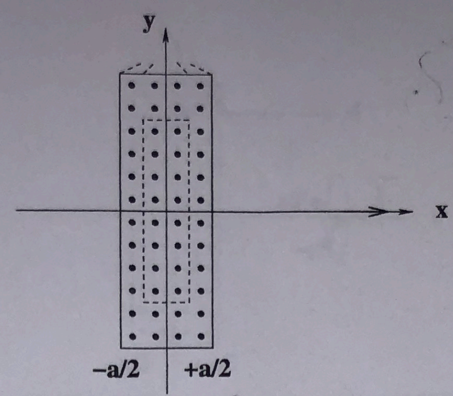
$\int \vec{B} \cdot d\vec{s} = \mu_0 I_{encl}$



$\int \vec{B} \cdot d\vec{s} = \mu_0 I_{encl}$

$B(y) = \mu_0 I_{encl}$

A non-uniform current described by the current density

$$\vec{J} = \vec{J}_0 \cos\left(\pi \frac{x}{a}\right)$$

flows out of the plane of the page through a conducting sheet that extends horizontally from $x = -\frac{a}{2}$ to $x = +\frac{a}{2}$ and vertically from $y = -\infty$ to $y = +\infty$.

- 1a) (10 pts) How much current will pass through a rectangular loop that extends horizontally from $-x$ to $+x$ and vertically from $-y$ to $+y$ if *i)* $x < \frac{a}{2}$ and *ii)* $x > \frac{a}{2}$

7

$I_{encl} = \int \vec{J} \cdot d\vec{A} = \int \vec{J}_0 \cos\left(\pi \frac{x}{a}\right) |d\vec{A}| \cancel{\cos\theta} \leftarrow \theta = 0$

$= |\vec{J}_0| \int_{-y}^{y}\int_{-x}^{x} \cos\left(\frac{\pi}{a} x\right) dx\, dy = |\vec{J}_0| (2y) \left(\frac{a}{\pi} \sin\left(\frac{\pi}{a} x\right)\right)\Big|_{-x}^{x}$

$= 2y J_0 \left(\frac{a}{\pi}\right)\left[\sin \frac{\pi x}{a} - \sin\left(-\frac{\pi x}{a}\right)\right] = \frac{2y J_0 a}{\pi}\left(2 \sin\left(\frac{\pi x}{a}\right)\right)$

$$\boxed{I_{encl} = \frac{4 J_0 y a \sin\left(\frac{\pi x}{a}\right)}{\pi}}$$ ✓

$x > a/2$?!

6

- 1b) (10 pts) Find the (vector) magnetic field for *all* points on the x-axis.

$\left(x < -\frac{a}{2}\right)$: $\int \vec{B} \cdot d\vec{s} = \mu_0 I_{encl}$

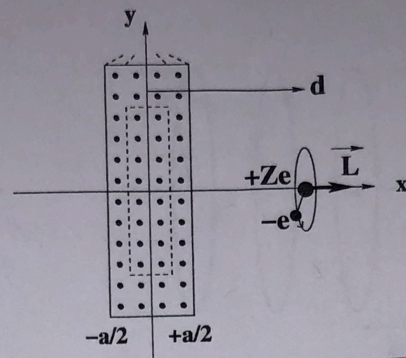
$-\hat{y}$ $2 \cdot B(2y) = \mu_0 I_{encl}$

$\left(x > \frac{a}{2}\right)$: $+\hat{y}$

$\left(-\frac{a}{2} < x < \frac{a}{2}\right)$: $\boxed{\vec{B} = \vec{0}}$ ✗

$$\boxed{\vec{B} = \frac{4\mu_0 a \sin\left(\frac{\pi x}{a}\right)}{\pi}(\pm\hat{y})}$$

$\gamma = \frac{\vec{\mu}}{\vec{\ell}}$



$\vec{\tau} = \vec{\mu} \times \vec{B}$

$L = I\vec{\omega}$ $\quad \omega = \frac{L}{I}$

$I = mr^2$ $\quad \omega = \frac{L}{mr^2}$

$|d\vec{\mu}| = N dI A$

* 8

- 1c) (10 pts) An electron (mass m, charge $-e$) orbits a nucleus (charge $+Ze$) located at the point $< d, 0, 0 >$. If the electron has an orbital angular momentum given by $\vec{L} = L\,\hat{x}$, find the magnitude and direction of i) the magnetic dipole moment associated with the electron's orbital motion and ii) the torque on the atom due to the magnetic field created by the current sheet.

$\vec{L} = L\hat{x}$

i) $d\vec{\mu} = N dI A (\hat{x})$

$dI = \frac{dq}{T} = \frac{dq}{\frac{2\pi}{\omega}} = \frac{\omega\, dq}{2\pi}$

$d\vec{\mu} = NA\left(\frac{\omega\, dq}{2\pi}\right)\hat{x}$

$= \cancel{N}\cancel{\pi r^2}\left(\frac{L}{m\cancel{r^2}}\right)\left(\frac{dq}{2\cancel{\pi}}\right)\hat{x}$ $\quad N = 1$

$d\vec{\mu} = \frac{L\, dq}{2m}\hat{x}$

oh..

$\boxed{\vec{\mu} = -\left(\frac{Le}{2m}\right)\hat{x}}$

ii) $\vec{\tau} = \vec{\mu} \times \vec{B}$

$= \left(\frac{Le}{2m}\right)\left(\frac{\cancel{4}2 \mu_0 a \sin\left(\frac{\pi x}{a}\right)}{\pi}\right)\sin\left(\frac{\pi}{2}\right)$

$\boxed{\vec{\tau} = \frac{2Le\mu_0 a \sin\left(\frac{\pi x}{a}\right)}{m\pi}\hat{\theta}}$

Should use $\vec{B}$ for $x > \pm \frac{a}{2}$

$$\frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}$$

$$\frac{\mu_0 I}{4\pi R^2}\int d\vec{l}$$

$$\frac{\mu_0 I}{\cancel{4\pi R^2}\,2}(\cancel{2\pi}R) = \frac{\mu_0 I}{2R}$$

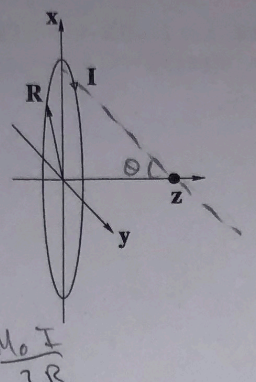


Figure 1

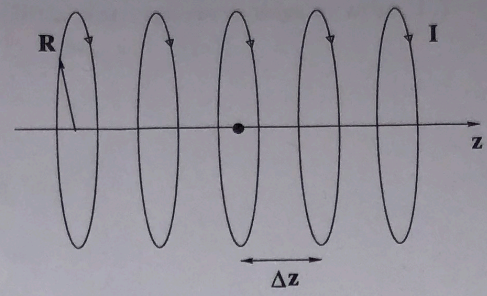


Figure 2

- 2a) (10 points) Figure 1 shows a circular conducting loop of radius R, situated in the x, y plane, centered on the origin. It carries a current I which is directed in the $+y$ direction at the top of the loop. Derive the magnetic field (magnitude and direction) for all points on the z-axis.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2} \qquad r = \sqrt{R^2+z^2} \qquad \cos\theta = \frac{z}{\sqrt{R^2+z^2}}$$

$$|d\vec{B}| = \frac{\mu_0}{4\pi}\frac{I}{r^2}|d\vec{l}||\hat{r}|\cos\theta$$

$$|\vec{B}| = \frac{\mu_0 I}{4\pi(R^2+z^2)^{3/2}}\int d\vec{s}$$

$$|\vec{B}| = \frac{\mu_0 I}{\cancel{4\pi}(R^2+z^2)^{3/2}}(\cancel{2\pi}R)$$

$$|\vec{B}| = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\hat{z}$$

- 2b) (5 points) In Figure 2, we find a collection of conducting loops (Radius R, oriented parallel to the x, y-plane, centered on the z-axis) that extend in steps of Δz from $-N\Delta z$ to $+N\Delta z$. Each carries a current I which is directed in the $+y$ direction at the top of the loop. Derive the magnetic field (magnitude and direction) at the center of the assembly.

2

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 + \vec{B}_5$$

$$\boxed{\vec{B} = \frac{\mu_0 I R^2}{2}\left[\frac{2}{(R^2+(2\Delta z)^2)^{3/2}} + \frac{2}{(R^2+\Delta z^2)^{3/2}} + \frac{1}{R^3}\right]}$$

↑ for the 5 shown

- 2c) (5 points) Use $2N\Delta z = L$ and $z = (index) * \Delta z$ to re-write your answer to the previous part in a form suggestive of the Riemann sum. (Hint: there are many ways to write '1'.)

- 2c) (10 points) Convert the Riemann sum to an integral and evaluate $\vec{B}$ at the center of this solenoid of finite length. Evaluate your answer (carefully) in the limit that $L \to \infty$ and show it yields the correct result.

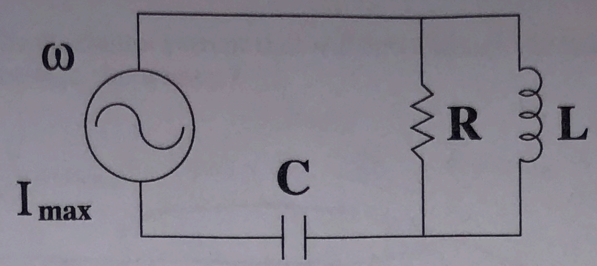
$\oint B \cdot ds = \mu_0 I_{encl}$

Will be equal to:

$B z = \mu_0 N I$

$B = \frac{\mu_0 N I}{z}$

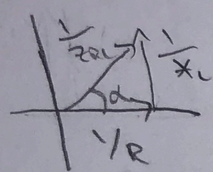
$\vec{B} = \mu_0 I \hat{z}$



3) A sinusoidally-varying **current** of amplitude I_{max} and angular frequency ω drives the RLC network shown above.

- 3a) (5 pts) Find the impedance of the RL combination.

$$\frac{1}{Z_{RL}} = \sqrt{\frac{1}{X_L^2} + \frac{1}{R^2}} = \sqrt{\frac{1}{(\omega L)^2} + \frac{1}{R^2}}$$



$$\boxed{Z_{RL} = \frac{1}{\sqrt{\left(\frac{1}{\omega L}\right)^2 + \frac{1}{R^2}}}}$$

voltage leads by α

$$\tan\alpha = \frac{1/X_L}{1/R} = \frac{R}{X_L} = \frac{R}{\omega L}$$

- 3b) (5 pts) Will the voltage across the RL combination lead or lag the current that passes through it? By how much?

ELI ICE

Voltage leads current

$$\boxed{\text{lead by } \alpha = \arctan\left(\frac{R}{\omega L}\right)}$$

ELI

- 3c) (10 pts) What is the maximum current that will flow through the inductor? What is the maximum current that will flow through the resistor?

Inductor 4

$\mathcal{E}_{max} = I_{max} Z \rightarrow I_{max} = \frac{\mathcal{E}_{max}}{Z}$

$\boxed{I_{L,max} = \frac{\mathcal{E}_{max}}{\omega L} \cos(\omega t - \frac{\pi}{2})}$

$\mathcal{E}_{max} = I_{max} Z_{RL}$

$\mathcal{E}_{max} = \frac{I_{max}}{\sqrt{(\frac{1}{\omega L})^2 + \frac{1}{R^2}}}$

Resistor

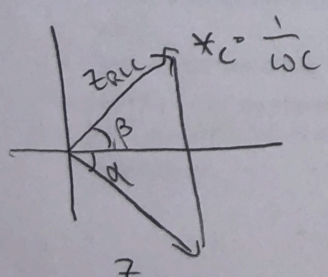
$\boxed{I_{R,max} = \frac{\mathcal{E}_{max}}{R} \cos(\omega t)}$

- 3d) (5 pts) Will $I_{C,max} = I_{L,max} + I_{R,max}$? Explain.

Yes, $I_{C,max} = I_{L,max} + I_{R,max}$ because the capacitor is in series with the parallel combination of the other two, and so the current through them must be equal.

1

- 3e) (5 pts) Draw a phasor diagram (in impedance space) to show how you would combine the capacitor with the *RL* combination to find the total impedance seen by the source. Label the contributions from the capacitor and the *RL* combination clearly, and make sure you clearly identify the relevant angles for anything that doesn't sit on an axis.



$Z_{RLC} \rightarrow$ total impedance of circuit

$X_C = \frac{1}{\omega C} \rightarrow$ reactance of capacitor

$Z_{RL} \rightarrow$ impedance of resistor-inductor parallel combination

$\alpha \rightarrow$ determined in part (b)

$\beta \rightarrow$ new angle (currently unknown but could be easily found) that Z_{RLC} makes with real axis.

3