1CS20 Final Exam

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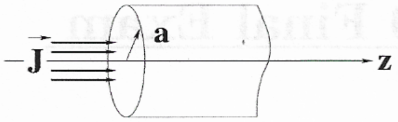
- The exam is open-book and open notes. You will probably do better to limit yourself to a single page of notes you prepared well in advance.
- **All work must be your own.** You are not allowed to collaborate with anyone else, you are not allowed to discuss the exam with anyone until all the exams have been submitted (after the close of the submissions window for the exam).
- You have **120 minutes** to complete the exam and more than sufficient time to scan the exam and upload it to GradeScope. The exam *must* be uploaded to GradeScope within the time alloted (that is, by the end of the 3-hour finals slot). We will only except submissions through GradeScope and will not accept any exam submitted after the submission window closes (CAE students must contact Corbin for instructions).
- **Given the limits of GradeScope, you must fit your work for each part into the space provided.** You may work on scratch paper, but you will not be able to upload the work you do on scratch paper, so it is essential that you copy your complete solution onto the exam form for final submission. We can only consider the work you submit on your exam form.
- **For full credit the grader must be able to follow your solution from first principles to your final answer.** ***There is a valid penalty for confusing the grader.***
- It is **YOUR** responsibility to make sure the exam is scanned correctly and uploaded before the end of the submission window. The graders may refuse to grade pages that are significantly blurred, solutions to problems that are not written in the correct place, pages submitted in landscape mode and/or work that is otherwise illegible - if any of this occurs, you may not receive *any* credit for the affected parts.
- Focus on the concepts involved in the problem, the tools to be used, and the set-up. If you get these right, all that's left is algebra.
- **Have Fun!**

The following must be signed before you submit your exam:

By my signature below, I hereby certify that all of the work on this exam was my own, that I did not collaborate with anyone else, nor did I discuss the exam with anyone while I was taking it.

Signature Cole Trammer



1) An electrical current described by the current density

$$\vec{J} = J_0(1 - 2\frac{r^2}{a^2})\,\hat{z}$$

flows in a cylindrical region of radius a aligned with and centered on the z-axis as shown.

- **1a) (10 points)** How much current is enclosed by a circular loop of radius r centered on the symmetry-axis of the cylinder, oriented perpendicular to the current-density? Consider both $r \le a$ and $r > a$.

$r \le a$: $I = \int_0^r \vec{J}\cdot d\vec{A} = \int_0^r J_0\left(1 - \frac{2r^2}{a^2}\right)(2\pi r\, dr) = 2\pi J_0 \int_0^r \left(r - \frac{2r^3}{a^2}\right) dr$

$= 2\pi J_0 \left[\frac{r^2}{2} - \frac{r^4}{2a^2}\right]_0^r = \boxed{\pi J_0 \left(r^2 - \frac{r^4}{a^2}\right)}$

$r > a$: $I = \int_0^a \vec{J}\cdot d\vec{A} = \pi J_0 (a^2 - a^2) = \boxed{0}$

- **1b) (5 points)** Find the (vector) magnetic field at a radial distance r from the symmetry-axis of the cylinder for points inside and outside the cylinder.

$r \le a$: $\oint \vec{B}\cdot d\vec{r} = \mu_0 I_{enc} \Leftrightarrow B(2\pi r) = \mu_0 \pi J_0 \left(r^2 - \frac{r^4}{a^2}\right) \Rightarrow B = \frac{\mu_0}{2}\left(r - \frac{r^3}{a^2}\right)$

$\boxed{\vec{B} = \frac{\mu_0}{2}\left(r - \frac{r^3}{a^2}\right)\hat{\phi}}$, where $\hat{\phi}$ is clockwise relative to $\vec{J}$

$r > a$: $\oint \vec{B}\cdot d\vec{r} = \mu_0 I_{enc} = 0 \Rightarrow \boxed{\vec{B} = 0}$

- 1c) (5 points) At what value of r does $\vec{J}$ change direction? At what value r does the net current through a loop of radius r change direction? At what value of r does $\vec{B}$ change direction? Discuss.

$\vec{J}$ changes direction when $\vec{J}=0 \Rightarrow 1 = \frac{2r^2}{a^2} \Rightarrow \frac{a^2}{2} = r^2 \Rightarrow r = \frac{a}{\sqrt{2}}$

The net current never changes direction, but goes to 0 at $r=a$

$\vec{B}$ changes direction when $\vec{B}=0 \Rightarrow r = \frac{r^3}{a^2} \Rightarrow r=a$, but $\vec{B}$ is always 0 at $r \geq a$, so $\vec{B}$ never changes direction.

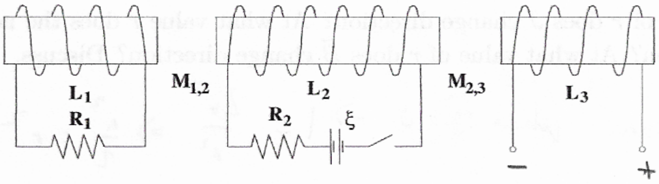
- 1d) (10 points) Find the energy per unit-length associated with the current.

$$u_B = \frac{1}{2\mu_0} B^2$$

$$\frac{1}{2\mu_0} \int_0^a \left(\frac{\mu_0}{4} \left(r - \frac{r^3}{a^2} \right) \right)^2 (2\pi r \, dr) = \frac{4\mu_0}{\pi} \int_0^a r \left(r^2 - \frac{2r^4}{a^2} + \frac{r^6}{a^4} \right) dr$$

$$= \frac{4\mu_0}{\pi} \left[\frac{r^4}{4} - \frac{2r^6}{3a^2} + \frac{r^8}{8a^4} \right]_0^a$$

$$= \frac{4\mu_0}{\pi} \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8} \right) a^4 = \boxed{\frac{\mu_0 a^4}{6\pi}}$$



2) All the values shown in the diagram above are known. The switch has been open a long time. It will be closed at $t = 0$.

- 2a) (10 points) Use one of Kirchoff's laws to find the current through and the electromotive-force across L_2 as a function of time elapsed after the switch was closed. Don't make any assumptions about the initial current in the circuit just yet, call it I_0 for now.

$$\xi - iR_2 - L_2 \frac{di}{dt} = 0 \Rightarrow \int_{I_0}^{i} \frac{di}{i - \frac{\xi}{R_2}} = \int_0^t -\frac{R_2}{L_2}\, dt$$

$$\Rightarrow \ln\left|i - \frac{\xi}{R_2}\right|\Bigg|_{I_0}^{i} = -\frac{R_2}{L_2} t \Rightarrow \frac{i - \frac{\xi}{R_2}}{I_0 - \frac{\xi}{R_2}} = e^{-\frac{R_2}{L_2} t}$$

$$\boxed{i = \frac{\xi}{R_2} + \left(I_0 - \frac{\xi}{R_2}\right) e^{-\frac{R_2}{L_2} t}}$$

- 2b) (5 points) Take a close look at the Kirchoff equation you wrote for the circuit and explain why I_0 has to be zero. It may be a good idea, at this point, to re-write your answers to part *a* with this new information taken into account.

Since the switch is open, there cannot be any current flow and $\frac{di}{dt}$ will be zero, but so will I_0

$$i = \frac{\xi}{R_2}\left(1 - e^{-\frac{R_1}{L_2} t}\right)$$

• 2c) (10 points) Find the potential difference across and the current through R_1 as a function of time elapsed after the switch was closed. In what direction does the current flow through R_1? Explain.

$\mathcal{E}_1 = -M_{1,2}\frac{di}{dt}$ $\mathcal{E}$ in L_1, voltage across R_1 must be equal

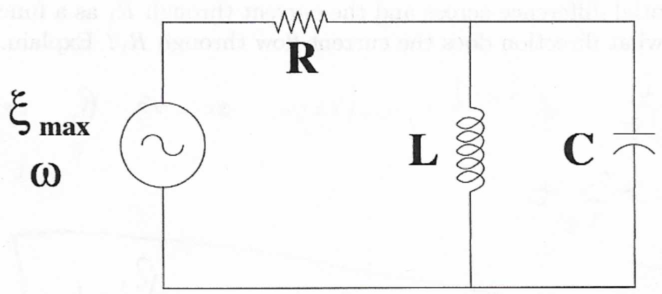
$\frac{di}{dt} = \frac{\mathcal{E}}{R_2}\cdot\frac{R_2}{L_2}e^{-\frac{R_2}{L_2}t}$

$\Rightarrow \boxed{\mathcal{E}_1 = -\frac{\mathcal{E}M_{2,1}}{L_2}e^{-\frac{R_2}{L_2}t}}$, $\boxed{I_1 = -\frac{\mathcal{E}M_{1,2}}{L_2R_1}e^{-\frac{R_2}{L_2}t}}$

The current flows counter clockwise through R_1, since it flows clockwise in R_2, and there is a negative sign.

• 2d) (5 points) After a long time, the switch is once again opened. At that instant, estimate the magnitude of the electromotive force across L_3 and mark the polarity of that electromotive force on the open terminals in the diagram.

$\mathcal{E}_{L_3} = -\frac{\mathcal{E}M_{2,3}}{L_2}e^{-\frac{R_2}{L_2}t}$, when $t\to\infty$, $\boxed{\mathcal{E}_3 = 0}$



3) The quantities shown in the diagram above are all known. When you answer the following questions, make sure your final answers are written in terms of ω (in other words, don't leave any X's lying around).

- 3a) (10 points) Find the impedance presented to the circuit by the L-C portion of the circuit.

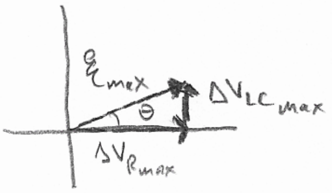
$$\frac{1}{Z} = \frac{1}{X_L} - \frac{1}{X_C} = \frac{1}{\omega L} - \omega C$$

$$\Rightarrow \boxed{Z_{LC} = \frac{1}{\frac{1}{\omega L} - \omega C}}$$

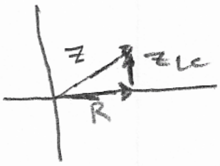
- 3b) (5 points) Under what conditions will the voltage across the L-C portion lead the current through the L-C portion of the circuit?

Voltage leads current in capacitors, so voltage will lead in the L-C portion when $X_C > X_L$ or $\frac{1}{\omega C} > \omega L$.

- **3c) (10 points)** How large is the amplitude of the current drawn out of the source?



$\div I_{max}$



$$Z = \sqrt{R^2 + (Z_{LC})^2} = \sqrt{R^2 + \frac{1}{\frac{1}{\omega L} - \omega C}}$$

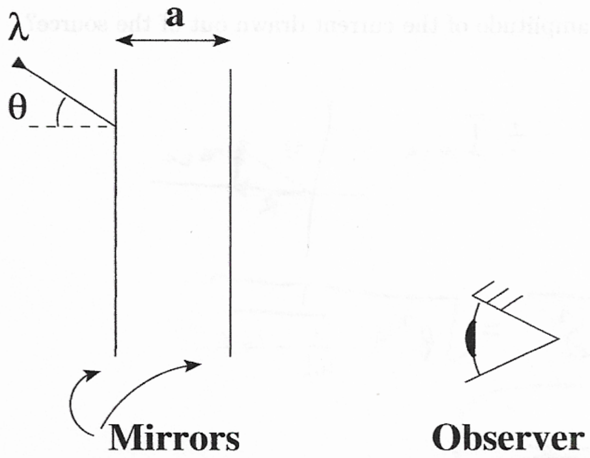
$$\boxed{I_{max} = \frac{\mathcal{E}_{max}}{\sqrt{R^2 + \frac{1}{\frac{1}{\omega L} - \omega L}}}}$$

- **3d) (5 points)** Under what conditions will the current drawn from the source lead the voltage delivered by the source? Explain.

$$\tan\theta = \frac{\frac{1}{\frac{1}{\omega L} - \omega C}}{R}$$

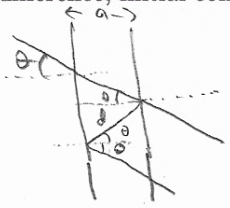
If $\tan\theta > 0$, $\theta > 0$, and the above reactance functions more like an inductor. Voltage leads current in capacitors,

so $$\frac{\frac{1}{\frac{1}{\omega L} - \omega C}}{R} < 0.$$



4) A Fabry-Perot interferometer consists of two parallel, half-silvered mirrors that are separated by a small distance a. Each of the mirrors transmits 50% of the incident light intensity and reflects the other 50% of the incident intensity. For the following, assume the angle of incidence is given by θ.

- 4a) (10 points) Derive the contributions to total phase difference at the location of the observer made by path difference, initial conditions and reflection. Be careful - the angles are not necessarily small.



$\Delta\theta_{ic} = 0$; $\Delta\theta_{ref} = 0$, since there is 2 reflections

$\Delta\theta_{path} = k\left(\frac{2a}{\cos\theta}\right)$

$\cos\theta = \frac{a}{d}$

$d = \frac{a}{\cos\theta}$

$$\boxed{\Delta\theta_{tot} = \frac{4\pi a}{\lambda\cos\theta}}$$

- 4b) (5 points) Under what conditions will the light observed have maximum intensity?

$\frac{4\pi a}{\lambda\cos\theta} = 2N\pi \Rightarrow \frac{4a}{\lambda} = 2N\cos\theta \Rightarrow \boxed{\cos\theta = \frac{2a}{N\lambda}}$

The cosine of the angle of incidence must be an interger multiple $\frac{2a}{\lambda}$

- 4c) (10 points) If light of wavelength λ_0, incident at an angle θ, produces a signal of maximum intensity at the location of the observer, how far from λ_0 will the next longer wavelength that produces a signal of maximum intensity at an angle of θ appear (that is, what is $\Delta\lambda$, if $\Delta\lambda \equiv \lambda_{next} - \lambda_0$)?

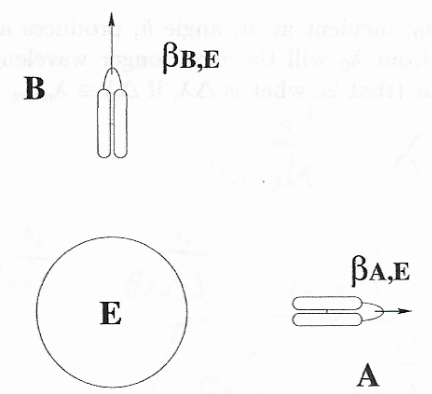
$$\cos\theta = \frac{2a}{N\lambda} \Rightarrow \lambda = \frac{2a}{N\cos\theta}$$

$$\lambda_0 = \frac{2a}{\cos\theta} \ ; \ \lambda_{next} = \frac{2a}{2\cos\theta} = \frac{a}{\cos\theta}$$

$$\boxed{\Delta\lambda = \frac{a}{\cos\theta} - \frac{2a}{\cos\theta}}$$

- 4d) (5 points) Suppose you wanted to design a filter to extract light of wavelength λ_0 from a mixture with other signals flowing down an optical fiber. Would you want $\Delta\lambda$ to be large or small? Explain. If one were to define a quality factor for a Fabry-Perot filter, would it be proportional or inversely proportional to $\Delta\lambda$?

You would want $\Delta\lambda$ to be large, and thus a filters quality would be proportional to $\Delta\lambda$. This way the filter would be not allow light with wavelengths close to λ_0 through.



5) Rocket A is moving with a velocity $(\beta_{A,E}\, C)\,\hat{x}$ relative to the Earth and Rocket B is moving with a velocity $(\beta_{B,E}\, C)\,\hat{y}$ relative to the Earth. Make sure your final answers to the following questions are in terms of the given information (no γ's).

- 5a) (5 points) A stick of length L_A lies along the x axis in Rocket A. How long is the stick in the Earth's frame? If the stick ages a year in Rocket A, how much has it aged in the Earth's frame?

$$L_{AE} = \frac{1}{\gamma_A} L_A = \boxed{L_A\sqrt{1-\beta_{A,E}^2}}$$

$$\Delta t_E = \frac{\Delta t}{\sqrt{1-\beta_{A,E}^2}} \text{, when } \Delta t = 1 \text{ year, } \boxed{\Delta t_E = \frac{1}{\sqrt{1-\beta_{A,E}^2}} \text{ years}}$$

- 5b) (10 points) It takes a bug a time Δt_A to walk along the stick (back-to-front) in (and relative to) Rocket A. How long did it take the bug in the Earth frame? How far did the bug travel in the Earth frame?

$$\begin{pmatrix} \Delta t_E \\ \Delta L_E \end{pmatrix} = \gamma_A \begin{pmatrix} 1 & \beta_{A,E} \\ \beta_{A,E} & 1 \end{pmatrix} \begin{pmatrix} \Delta t_A \\ L_A \end{pmatrix}$$

$$\Rightarrow \Delta t_E = \gamma_A(\Delta t_A + \beta_{A,E} L_A) = \boxed{\frac{1}{\sqrt{1-\beta_{A,E}^2}}(\Delta t_A + \beta_{A,E} L_A)}$$

$$\Delta L_E = \gamma_A(\beta_{A,E} t_A + L_A) = \boxed{\frac{1}{\sqrt{1-\beta_{A,E}^2}}(\beta_{A,E} t_A + L_A)}$$

- 5c) (10 points) How fast is Rocket B moving in Rocket A's frame?

Rocket B is moving perpendicular to Rocket A, realative to Earth

$$\frac{v_{B,A\perp}}{c} = \frac{\beta_{B,E}}{\gamma_A(1+\beta_{A,E}\beta_{B,E})}$$

$$\Rightarrow v_{B,A_\perp} = \frac{\beta_{B,E}c}{\frac{1}{\sqrt{1-\beta_{A,E}^2}}(1+0)} = \beta_{B,E}\sqrt{1-\beta_{A,E}^2}\,\beta_{B,E}c$$

$$\frac{v_{B,A\parallel}}{c} = \frac{\beta_{B,E}+0}{1+0} \Rightarrow v_{B,A_\parallel} = \beta_{A,E}c$$

$$\boxed{v_{B,A} = \sqrt{c^2\left(\beta_{A,E}^2 + (1-\beta_{A,E}^2)\beta_{B,E}^2\right)}}$$

- 5d) (5 points) What is the significance of $\beta = 0$ and $\beta = 1$? Evaluate your answer to part c in the limits $\beta_{B,E} = 0$ and $\beta_{B,E} = 1$ and discuss...

$\beta=0$ means there is no velocity, $\beta=1$ means a velocity of the speed of light

$\beta_{B,E}=0 \Rightarrow v_{B,A} = \sqrt{c^2(\beta_{A,E}^2+0)} = \beta_{A,E}c$, which makes complete sense, this is just the velocity of rocket A relative to the Earth.

$\beta_{B,E}=1 \Rightarrow v_{B,A} = \sqrt{c^2(\beta_{A,E}^2+(1-\beta_{A,E}^2)1)} = c$, so the speed of light is the same in all frames.