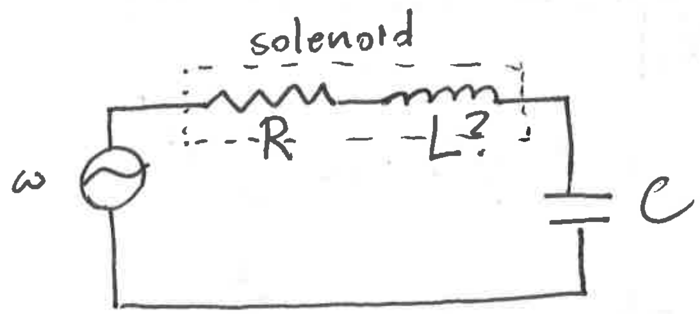
Problem 1. In the lab, you encounter a solenoid with unknown inductance. You measure the solenoid with an ohmmeter and find it has internal resistance R. To determine its inductance, you connect the solenoid in series with a known capacitor C and an AC signal generator (AC voltage source) set to frequency ω.

(6) **(a)** If the current amplitude is I_0, find the voltage amplitude $|V_C|$ across the capacitor.

(6) **(b)** With the same current, find the total voltage amplitude $|V_{tot}|$ across both the solenoid and capacitor in terms of the unknown inductance L and the known circuit parameters.

(6) **(c)** Suppose you don't know the current amplitude, but you measure the two voltage amplitudes in (a) and (b). Find the inductance L of the solenoid using these measurements and the known parameters.



(a) $\tilde{V} = \tilde{I}\tilde{Z}$ $\quad \tilde{Z}_c = -\frac{i}{\omega C} \rightarrow |\tilde{Z}_c| = \frac{1}{\omega C}$

$$|V_c| = I_0 |\tilde{Z}_c|$$

$$= \boxed{I_0/\omega C}$$

(b) series RLC circuit

$$|\tilde{Z}_{tot}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$|V_{tot}| = I_0 |\tilde{Z}_{tot}| = \boxed{I_0 \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

(c) Take

$$\frac{|V_{tot}|}{|V_c|} = \frac{I_0\sqrt{R^2+(\omega L - 1/\omega C)^2}}{I_0/(\omega C)} = \omega C\sqrt{R^2+\left(\omega L - \frac{1}{\omega C}\right)^2}$$

and solve for L. (Setup worth most of points, but some points for correct algebra.)

Let $\alpha = \dfrac{|V_{tot}|}{|V_c|} = \omega C\sqrt{R^2+(\omega L - 1/\omega C)^2}$

$$\rightarrow \alpha^2 = \omega^2 C^2\left(R^2+\left(\omega L - \frac{1}{\omega C}\right)^2\right)$$

$$\frac{\alpha^2}{\omega^2 C^2} - R^2 = \left(\omega L - \frac{1}{\omega C}\right)^2$$

$$\omega L - \frac{1}{\omega C} = \pm\sqrt{\frac{\alpha^2}{\omega^2 C^2} - R^2}$$

$$\boxed{L = \frac{1}{\omega^2 C} \pm \frac{1}{\omega}\sqrt{\frac{1}{\omega^2 C^2}\frac{|V_{tot}|^2}{|V_c|^2} - R^2}}$$

Problem 2. The magnetic field of a monochromatic, plane EM wave in vacuum is

$$\mathbf{B}_{\text{inc}}(x,t) = B_0\,\hat{\mathbf{y}}\,\cos(kx-\omega t) \tag{1}$$

(6) **(a)** Find the electric field $\mathbf{E}_{\text{inc}}$ of this wave.

(6) **(b)** Find the Poynting vector $\mathbf{S}_{\text{inc}}$ for this wave.

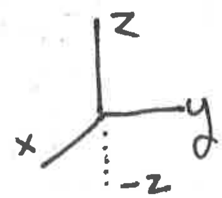
(6) **(c)** Suppose this wave is totally reflected by a thin sheet of a perfect conductor occupying the yz plane at $x = 0$. What is the force per unit area on this sheet due to the EM field?

(6) **(d)** When the wave reflects from the conductor at $x = 0$, a reflected plane wave traveling opposite the direction of the incident wave is formed. The total electric field $\mathbf{E}_{\text{tot}}$ is a superposition of the incident and reflected fields, $\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$. Take for granted that **the total electric field parallel to the surface of a perfect conductor is zero,** and find the reflected field $\mathbf{E}_{\text{ref}}$.

$\vec{B}_0 = \frac{1}{c}\hat{k}\times\vec{E}_0$ | $\cos(kx-\omega t) \rightarrow \vec{k}$ points in $+x$-direction

(a) Magnitude: $E_0 = cB_0$

Direction: $\hat{y} = \hat{x}\times\hat{E}_0$



by right-hand rule,

$\hat{x}\times(-\hat{z}) = \hat{y}$,

so $\hat{E}_0 = -\hat{z}$

$$\boxed{\vec{E}_{inc} = -cB_0\,\hat{z}\,\cos(kx-\omega t)}$$

(b)

$$\vec{S}_{inc} = \frac{1}{\mu_0}\vec{E}_{inc}\times\vec{B}_{inc}$$

$$= \boxed{\frac{1}{\mu_0}cB_0^2\,\hat{x}\,\cos^2(kx-\omega t)}$$

(c) force per area = pressure

$$\mathcal{P}_{ref} = 2\frac{\langle S\rangle_T}{c}$$

$$= \frac{2}{c}\frac{cB_0^2}{\mu_0}\underbrace{\langle\cos^2(kx-\omega t)\rangle_T}_{=1/2}$$

$$= \frac{2B_0^2}{\mu_0}\cdot\frac{1}{2}$$

$$= \boxed{\frac{B_0^2}{\mu_0}}$$

(d) $\vec{E}_{ref}$ travels in $-x$ direction. We know

$$\boxed{\vec{E}_{ref} = \vec{E}_{ref,0}\cos(kx+\omega t)}$$

What is $\vec{E}_{ref,0}$? ($\vec{E}_{r,0}$ for brevity)

$$\vec{E}_{tot} = \vec{E}_{i,0}\cos(kx-\omega t) + \vec{E}_{r,0}\cos(kx+\omega t)$$

We know $\vec{E}_{tot}$ parallel to conductor, since $\vec{E}$ must be perpendicular to $\hat{x}$. So at conductor $(x=0)$

$$\vec{E}_{tot}(x=0) = \vec{E}_{i,0}\cos(-\omega t) + \vec{E}_{r,0}\cos\omega t = 0$$

$$\Rightarrow \vec{E}_{r,0} = -\vec{E}_{i,0}$$

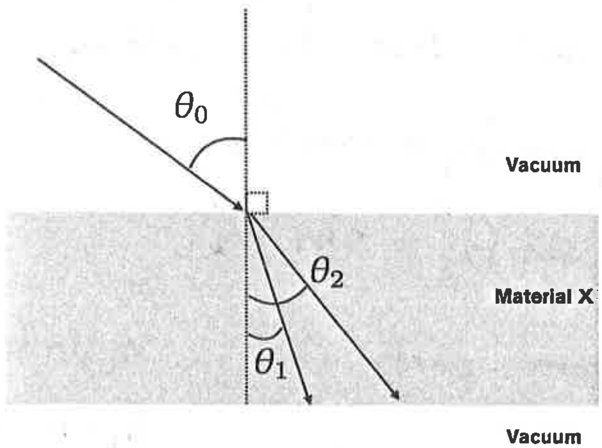
$$= -(-cB_0\hat{z})$$

$$= \boxed{cB_0\hat{z}}$$

Problem 3. Two monochromatic light waves of different wavelengths λ_1 and λ_2 in vacuum are both incident at angle θ_0 on rectangular prism of Material X. Material X is dispersive, and its index of refraction depends on the vacuum wavelength λ as

$$n(\lambda) = \frac{3}{2} + \frac{X}{\lambda^2}, \tag{2}$$

with $X > 0$ an *unknown* positive constant. Because n depends on λ, the two waves are refracted at different angles θ_1 and θ_2 in Material X, with $\theta_2 > \theta_1$.



(6) **(a)** Use Snell's law to find the values of the index of refraction $n_1 = n(\lambda_1)$ and $n_2 = n(\lambda_2)$ in terms of the incident and refracted angles.

For the two wavelengths, λ_1 and λ_2:

(4) **(b)** Which wavelength is larger?

(4) **(c)** Which wavelength of light travels faster in Material X?

(4) **(d)** In the figure above, which of the wavelengths, if any, will undergo total internal reflection at the bottom surface of the prism?

(a) Snell's law :

$\sin\theta_0 = n_1 \sin\theta_1 \rightarrow \boxed{n_1 = \frac{\sin\theta_0}{\sin\theta_1}}$

$\sin\theta_0 = n_2 \sin\theta_2 \rightarrow \boxed{n_2 = \frac{\sin\theta_0}{\sin\theta_2}}$

(b) Since $\theta_2 > \theta_1$, $n_1 > n_2$

since $n \sim \frac{1}{\lambda^2}$, larger $n \sim$ smaller λ

$\Rightarrow \boxed{\lambda_2 > \lambda_1}$

(c) $n_1 > n_2$

$$V_1 = \frac{c}{n_1} \quad , \quad V_2 = \frac{c}{n_2}$$

so $\boxed{V_1 < V_2}$

(d) Apply Snell's law to bottom surface:

$$n_1 \sin\theta_1 = \sin\theta_1'$$
$$n_2 \sin\theta_2 = \sin\theta_2'$$

But from part (a), $n_1 \sin\theta_1 = \sin\theta_0$
$n_2 \sin\theta_2 = \sin\theta_0$

so outgoing angle for both rays is $\theta_0 < \frac{\pi}{2}$.

$\Longrightarrow$ $\boxed{\text{no T.I.R. for either wavelength}}$