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Physics 1BH – Prof. J. Rosenzweig – Winter 2017
Midterm 1
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Use only the paper provided for you. Show all of your work for full credit. Write your name on each sheet of paper in your answers, then staple all together in order. You have 1 hour and 50 minutes to complete this exam. You are permitted one sheet of paper as notes, with writing on both sides.

1. Consider a scalar potential that is a function of only x and y,

$$\phi(x,y) = V\exp\left[-\frac{(x^2+y^2)}{2a^2}\right],$$

where V and a are constants with the appropriate units.

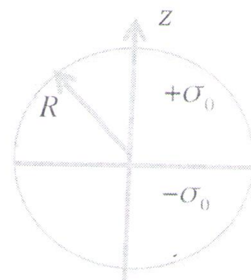
31 (a) (10 pts) Find the electric field associated with this potential from the gradient of this function in Cartesian coordinates?

(b) (10 pts) What is the divergence of the electric field found in part (a)

(c) (5 pts) The contours of constant f are circles in the (x,y) plane. Draw some arrows indicating the direction of the gradient relative to these contours.

(d) (10 pts) What is the charge density ρ associated with this potential?

14/30 2. Consider a surface charge distribution on a spherical shell of radius R. It is split into two components, one above the z=0 plane which has surface charge density $+\sigma_0$ and one below the $-\sigma_0$, as shown:



9 (a) (10 pts) Find the electric field at the origin.

5 (b) (10 pts) Now find by direct integration the potential associated with this distribution on the z axis for $z>R$.

0 (c) (10 pts) What is the dipole moment associated with this potential?

3. Consider a *spherically* symmetric charge distribution, of the form

$$\rho(r) = a\frac{\rho_0}{r},$$

where V and a are constants with the appropriate units.

(a) (10 pts) Using Gauss' law, find the electric field associated with this charge distribution.

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(b) (10 pts) What can you say about the electric field as the origin is approached (*i.e.* why is there uncertainty in the direction of field at r=0)?

(c) (10 pts) Setting the potential at the origin to zero, what is the potential in all space? (5 pts) Can you explain why it must diverge as $r \to \infty$?

1a. $E = -\nabla \phi = -\nabla V \exp\left(\frac{-(x^2+y^2)}{2a^2}\right)$

$= \hat{x}\frac{-\partial}{\partial x} V\exp\left(\frac{-(x^2+y^2)}{2a^2}\right) + \hat{y}\frac{-\partial}{\partial y} V\exp\left(\frac{-(x^2+y^2)}{2a^2}\right)$

$= \hat{x}\frac{2x}{2a^2} V e^{\frac{-(x^2+y^2)}{2a^2}} + \hat{y}\frac{2y}{2a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}$

$= \boxed{\frac{x}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}\hat{x} + \frac{y}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}\hat{y}}$ ✓ + 10

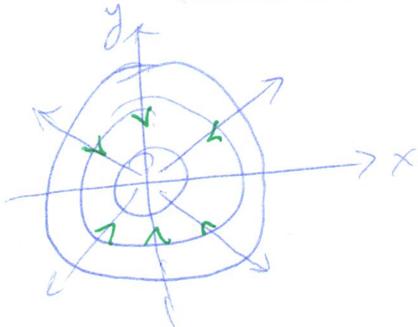
1b. $\nabla \cdot E = \frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} = \frac{1}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}} + \frac{-x^2}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}$

+5

$+ \frac{1}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}} + \frac{-y^2}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}$

+3 $= \boxed{\frac{1}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}\left(2 - \frac{x^2 - y^2}{a^2}\right)}$ careful, these units don't match (scalar + length)

1c.



+3 ($\nabla\phi$, not $-\nabla\phi$)

1d. By Gauss' Law:

$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ so $\frac{\rho}{\varepsilon_0}$ = result in 1b

$\frac{\rho(x,y)}{\varepsilon_0} = \frac{1}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}(2-x-y)$

$\boxed{\rho(x,y) = \frac{\varepsilon_0}{a^2} V e^{\frac{-(x^2+y^2)}{2a^2}}(2-x-y)}$

E.C.F.

+10

2a. For a hemispherical shell:

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$$E_{ring} = \frac{Q_{ring}\, z}{4\pi\varepsilon_0 \left(\sqrt{z^2+a^2}\right)^3}$$

For $E_{hem} = \int dE_{ring}$

But for a hemisphere $r = \sqrt{z^2+a^2}$ is constant so

$$E_{hem} = \int \frac{dQ_{ring}\, z}{4\pi\varepsilon_0 r^3}$$

$$dQ_{ring} = \underbrace{(2\pi r \sin\theta)}_{\text{circumference}} \underbrace{(r\, d\theta)}_{\text{thickness}} \sigma \qquad z = r\cos\theta$$

volume element is $dA = \underline{r}^2 \sin\theta \, d\theta \, d\phi$

$$E_z = \int_0^{\frac{\pi}{2}} \frac{2\cancel{\pi}\cancel{r}\sin\theta\, \cancel{r}\, d\theta\, \sigma\, r\cos\theta}{4\cancel{\pi}\varepsilon_0 \cancel{r^3}}$$

$$= \frac{r\sigma}{4\varepsilon_0}\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta = \frac{r\sigma}{4\varepsilon_0}\left(\frac{-\cos 2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{r\sigma}{4\varepsilon_0}(1) = \frac{r\sigma}{4\varepsilon_0}$$

Forgot we used R in provided diagram so

$$\frac{R\sigma}{4\varepsilon_0}$$

So for a positive hemisphere and negative hemisphere:

(+) one contributes $\frac{R\sigma}{4\varepsilon_0}$ downwards

(−) one attributes $\frac{R\sigma}{4\varepsilon_0}$ downwards too

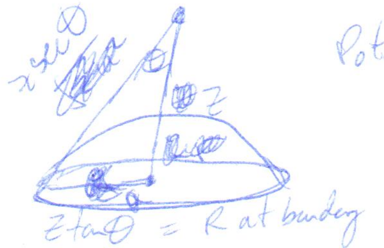
So $\boxed{E = \frac{R\sigma}{2\varepsilon_0} \text{ downwards } \left(\text{or } -\frac{R\sigma}{2\varepsilon_0}\hat{z}\right)}$

close, extra factor of R

2b. ~~Potential ring~~ = ~~$\oint E_{ry}\, dz$~~ = ~~$\int \frac{Q_{ry}\, z}{4\pi\varepsilon_0 (z^2+a^2)^3}\, dz$~~ =

5/10

= ~~$\oint E_{ry}\, dz = \int \frac{Q_{ry}\, z}{4\pi\varepsilon_0 \sqrt{z^2+a^2}}\, dz = \frac{Q_{ry}}{4\pi\varepsilon_0 (z^2+a^2)^{1/2}}$~~



$$\text{Potential ring} = -\int E_{ry}\, dz = -\int \frac{Q_{ry}\, z}{4\pi\varepsilon_0 \sqrt{z^2+a^2}^{\,3}}\, dz$$

not right expression for general $\vec{E}$ on z-axis

$$= \frac{Q_{ry}}{4\pi\varepsilon_0 (z^2+a^2)^{1/2}}$$

$$\phi_{hemi} = \int d\phi_{ry} = \int \frac{dQ_{ry}}{4\pi\varepsilon_0 (z^2+a^2)^{1/2}}$$

$$dQ_{ry} = \underbrace{(2\pi z\tan\theta)}_{\text{circumference}}(z\sec\theta\, d\theta)\,\sigma$$

how?

$a = z\tan\theta$

$$\phi_{hemi} = \int \frac{2\pi z\tan\theta\, z\sec\theta\, d\theta\, \sigma}{4\pi\varepsilon_0 (z^2+z^2\tan^2\theta)^{1/2}}$$

$z^2+z^2\tan^2\theta = z^2\sec^2\theta$

$$= \frac{\sigma}{2\varepsilon_0}\int \frac{z^2\tan\theta\sec\theta}{z\sec\theta}\, d\theta$$

$$= \frac{z\sigma}{2\varepsilon_0}\int_0^{\arctan\frac{R}{z}} \tan\theta\, d\theta$$

I don't know what this integrates to, but final answer is:

$2\phi_{hemi} - \phi_{shell}$ b/c of superposition

$$\text{so } \boxed{2\left(\frac{z\sigma}{2\varepsilon_0}\int_0^{\arctan\frac{R}{z}} \tan\theta\, d\theta\right) - \frac{\sigma(4\pi R^2)}{4\pi\varepsilon_0 z}}$$

$\frac{q}{4\pi\varepsilon_0 r}$

2c. Not sure what the dipole moment is but could probably find it I could evaluate the above...

0 need $z \to \infty$

3a Gauss' Law:

$$\oint_{\text{sfce}} E \cdot dA = \iiint_{\text{volume}} \frac{\rho}{\varepsilon_0} dV \qquad a\rho_0 r\sin\theta$$

$$4\pi r^2 (E) = \iiint_{0\,0\,0}^{2\pi\,\pi\,r} a\frac{\rho_0}{\cancel{r}} r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$4\pi r^2 (E) = (2\pi)(\cancel{2})\, a \frac{\rho_0 r^2}{\cancel{2}}$$

$$4\pi r^2 E = 2\pi a \rho_0 r^2$$

-1 (+9)

$$E = \frac{2\pi a \rho_0 r^2}{2\, 4\pi r^2} = \boxed{\frac{a\rho_0}{\varepsilon_0 2}\hat{r}}$$ for $r < R$ where R is radius of sphere

~~3b As ... $\rho(r)$ ... approaches infinity~~

3a cont. Outside of sphere though

No sphere, it's everywhere.

$$\oint E \cdot dA = \iiint \frac{\rho}{\varepsilon_0} dV$$

$$4\pi r^2 (E) = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} a\frac{\rho_0}{r} r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$4\pi r^2 (E) = (2\pi)(2)\, a \frac{\rho_0 R^2}{2}$$

$$E = \boxed{\frac{a\rho_0 R^2}{2r^2}\hat{r}}$$ for $r > R$

3b. As $r \to 0$, $\rho(r) \to \infty$ b/c the $\frac{1}{r}$ portion of ρ runs to infinity. This is problematic b/c we can't determine what the electric field should be nearly on top of an infinitely densely charged piece of space. +10

3c. $E = \begin{cases} \frac{a\rho_0}{2}\hat{r} & \text{inside} \\ \frac{a\rho_0 R^2}{2r^2}\hat{r} & \text{outside} \end{cases}$ $\quad \phi(r_2) - \phi(r_1) = -\int_{r_1}^{r_2} E \cdot dr$ b/c electric field is only radial

$$\text{inside} = -\int \frac{a\rho_0}{2} dr = -\frac{a\rho_0 r}{2\varepsilon_0}\Big|_{r_1}^{r_2}$$ +5

~~outside~~ $= \int \frac{a\rho_0 R^2}{2r^2} dr = \frac{a\rho_0 R^2}{2r}\Big|_{r_1}^{r_2}$

Potential is measured w.r.t. infinity so: wrt $r=0$ as stated in problem

for $r \geq R$ $\quad \frac{a\rho_0 R^2}{2r}\Big|_{\infty}^{r} = \frac{a\rho_0 R^2}{2r} - 0$

for $r < R$ $\quad -\frac{a\rho_0}{2} r\Big|_{R}^{r} + \frac{a\rho_0 R^2}{r}\Big|_{\infty}^{R} = -\frac{a\rho_0}{2} r + \frac{a\rho_0}{2} R + \frac{a\rho_0}{2} R - 0 \rightarrow$

3c
cont. $\phi(0)$ inside $= -\frac{a\rho_0}{2} r + a\rho_0 R$

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So $\phi(r) = \begin{cases} -\frac{a\rho_0}{2} r + a\rho_0 R & \text{inside } (r<R) \\ \frac{a\rho_0 R^2}{2r} & \text{outside } (r \geq R) \end{cases}$

normalize so $\phi(0) = 0$

$\phi(r) = \begin{cases} -\frac{a\rho_0 r}{2} \quad +3 & r<R \\ \cancel{\frac{a\rho_0 R^2}{2r} - a\rho_0 R} & r \geq R \end{cases}$

Diverges b/c of the weirdness at the center. Normalizing potential by setting it to 0 makes $r \to \infty$ behave weirdly instead. X

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