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MIDTERM #1
Physics 1BH
Prof. David Saltzberg
January 28, 2016

Time: 50 minutes. Closed Notes. Closed Book. Allowed the standard "cheat sheet". Calculators are allowed. Show your work.

If a problem is confusing or ambiguous, notify the professor. Clarifications will be written on the blackboard. Check the board.

There are 8 pages including this cover sheet. Make sure you have them all. Extra workspace is given and extra paper is at the front of the room.

Problem	Points
1	33 /33
2	33 /33
3	34 /34
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TOTAL	100 /100

1) An electric potential is given by $\varphi(r) = \varphi_0(r/a)^3 \exp(-r/a)$, where r is the distance from the z axis. φ_0 and a are constants. This problem has three parts on three pages:

A) What is the electric field, **E**, corresponding to this potential?

$$\vec{E} = -\vec{\nabla}\varphi$$

In cylindrical coordinates only E_r is non-zero for this $\varphi(r)$

$$\vec{E} = -\frac{\partial}{\partial r}\varphi(r)\,\hat{r}$$

$$= -\frac{\varphi_0}{a^3}\left[3r^2 e^{-r/a} + \frac{r^3 e^{-r/a}}{(-a)}\right]\hat{r}$$

$$= -\frac{\varphi_0}{a^3}\left[\left(3r^2 - \frac{r^3}{a}\right)e^{-r/a}\right]\hat{r}$$

$$\boxed{\vec{E} = -\frac{\varphi_0 r^2 e^{-r/a}}{a^3}\left(3 - \frac{r}{a}\right)\hat{r}}$$

B) What is the charge distribution that created this potential?

$$\rho = -\epsilon_0 \nabla^2 \varphi(r)$$

$$\rho = -\epsilon_0 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi(r)}{\partial r}\right)$$

Note, we are using cylindrical coordinates

For those of you that used $\frac{\rho}{\epsilon_0} = \vec{\nabla}\cdot\vec{E}$ that's fine but $\vec{\nabla}\cdot\vec{E}$ is [redacted] $\neq \frac{\partial E_r}{\partial r} + \dots$

$$= \frac{1}{r}\left(\frac{\partial (rE_r)}{\partial r}\right)$$

$$\rho = -\epsilon_0 \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\varphi_0 \frac{r^3}{a^3} e^{-r/a}\right)\right]$$

$$= -\frac{\epsilon_0 \varphi_0}{a^3}\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(3r^2 + r^3\left(-\frac{1}{a}\right)\right)e^{-r/a}\right]$$

$$= -\frac{\epsilon_0 \varphi_0}{a^3}\frac{1}{r}\left[9r^2 - \frac{4r^3}{a} - \left(\frac{1}{a}\right)\left(3r^3 - \frac{r^4}{a}\right)\right]$$

$$= -\frac{\epsilon_0 \varphi_0}{a^3}\frac{1}{r}\left[9r^2 - \frac{7r^3}{a} + \frac{r^4}{a^2}\right]$$

$$\rho(r) = -\frac{\epsilon_0 \varphi_0}{a^3}\left[9r - \frac{7r^2}{a} + \frac{r^3}{a^2}\right]$$

C) Is it possible to modify this $\varphi(r)$ so that **curl E** is non-zero? If so, give an example. If not, explain why not.

No, $\vec{E} = -\vec{\nabla}\varphi(r)$. Any function that is a gradiet of a scalar has a curl of $\emptyset$.

$\nabla \times (\nabla \varphi) = 0$ always

Also $\oint_C E \cdot dl = 0$ (for electrostatics) so

$\oint_S (\nabla \times E) \, dA = 0$ $\quad \vec{\nabla} \times \vec{E} = 0$ always

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Some of you interpreted $\varphi(r)$ to be constraned to be a function of r only. That made the question more specific-case but I gave you credit.

N.B. even if $\varphi = \varphi(r, \theta, z)$, $\nabla \times E = 0$ no matter what (static case)

2) In an atomic bomb, a plutonium nucleus (Z=94, A=240, where Z is the atomic number and A is the atomic mass.) will split into two smaller nuclei to release energy. Essentially all of the energy released comes from Coulomb's Law which causes the two positive fragments to repel to infinite separation.

Let us approximate the nuclei by spherical <u>shells</u> of charge with net charge $Q=Ze$ and radius R=(1.25 fm) $\sqrt[3]{A}$, (The latter formula makes sense for a ball that contains A neutrons+protons which are each of order ~1 fm in radius.)

(In reality the nucleus is more like a uniform sphere of charge, but that requires more calculation—as we did on the board for our model electron--and does not change the answer by more than about 2.)

Suppose the plutonium breaks up into two equal-sized nuclei (Z=47,A=120). How much energy is released by 1 kg of splitting plutonium? Compare this nuclear reaction to a strong chemical reaction, the explosion of T.N.T., for which the same mass (1kg) releases 4 MJ.

No working models, please.

Energy of bomb= $U_{before} - U_{after}$

For spherical shell $U = \frac{1}{2} Q V$ all at same radius

$$E = \Delta U = \frac{1}{2}\left[\frac{(94e)k(94e)}{(1.25\,fm)(240)^{1/3}} - 2\cdot\frac{(47e)k(47e)}{(1.25fm)(120)^{1/3}}\right]$$

$$= \frac{1}{2}\frac{ke^2}{(1.25\times10^{-15})}\left[\frac{94^2}{(240)^{1/3}} - 2\cdot\frac{47^2}{120^{1/3}}\right]$$

$$= \frac{(9\times10^9)(1.6\times10^{-19})^2}{2(1.25\times10^{-15})}\left[1422-896\right]$$

$$\boxed{E = 4.0\times10^{-11}\ J/atom}$$

— over —

[Extra space]

$$E_{\text{in 1kg}} = \left(4.0\times10^{-11}\frac{J}{\text{atom}}\right)(1000g)\left(\frac{\text{mole}}{240g}\right)\left(6\times10^{23}\frac{\text{atoms}}{\text{mole}}\right)$$

$$= 1.0\times10^{14} J$$

1kg TNT = 4 MJ
1 ton = 1000 kg
} 4×10^{9} J/ton of TNT

$E_{\text{bomb 1kg plutonium}}$ = 25,000 tons TNT !
= 25 kilotons

Full answer is
~ 20,000 tons

Note Why is $U = \frac{1}{2} Q \varphi$ and not $Q\varphi$?

(φ: energy per charge)

$U = Q\varphi$ if φ is provided by the fields of *other* charges.

Here U is from the self-assembly of Q

$$U = \int_0^Q \varphi(q')\, dq'$$

$$= \int_0^Q \frac{q'\, dq'}{4\pi\epsilon_0 R'}$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0 R}$$

$$U = \frac{1}{2} Q \varphi$$

(I did not take points off for missing this factor-of-two)

3) A spherical insulator extends from the origin out to a radius a. Beyond radius a is free space. How much work is required to assemble a charge distribution given by $\rho(r) = \rho_0({}^{r}/_{a})^3$?

see attached sol'n from Michael

MT #1

#3) $\rho(r) = \rho_0 \left(\frac{r}{a}\right)^3$ for $r \le a$, $\rho(r) = 0$ for $r > a$.

2 ways: 1) Gauss to find $\vec{E}$, then $U = \frac{\varepsilon_0}{2} \int_{\text{all space}} |\vec{E}|^2 \, dV$

2) Gauss to find $\vec{E}$, then find ϕ, then $U = \frac{1}{2} \int \rho \phi \, dV$

Note that this integral will only be over the sphere, since $\rho = 0$ outside the sphere.

Method #1):

Gauss $\Rightarrow$ $4\pi r^2 E_{in} = \frac{Q_{enc}}{\varepsilon_0}$ (E_{in}: E inside sphere) $\qquad dV = (r')^2 \sin\theta \, dr' \, d\theta \, d\phi$

where, for $r \le a$, $Q_{enc}(r) = \int_0^r \rho \, dV = \int_0^r \rho_0 \frac{(r')^3}{a^3} (r')^2 \, dr' \underbrace{\int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi}_{4\pi}$

(Q_{enc} = Q enclosed inside radius r ($\le a$))

$= \frac{4\pi\rho_0}{a^3} \int_0^r (r')^5 \, dr' = \frac{4\pi\rho_0}{6a^3} r^6$.

Thus $4\pi r^2 E_{in} = \frac{1}{\varepsilon_0} \frac{4\pi\rho_0}{6a^3} r^6 \Rightarrow \boxed{E_{in} = \frac{\rho_0}{6\varepsilon_0 a^3} r^4}$ (units ✓)

for E_{out}, we have Gauss $\Rightarrow$ $4\pi r^2 E_{out} = \frac{Q_{TOT}}{\varepsilon_0}$ ← total charge of sphere.

where $Q_{TOT} = Q_{enc}(r=a) = \frac{4\pi\rho_0 a^6}{6a^3} = \frac{4\pi\rho_0}{6} a^3$

Thus $\boxed{E_{out} = \frac{\rho_0 a^3}{6\varepsilon_0 r^2}}$

→

Method #2: $\vec{E}_{out} = \frac{\rho_0 a^3}{6\varepsilon_0 r^2}\hat{r}$ $\qquad$ $\vec{E}_{in} = \frac{\rho_0}{6\varepsilon_0 a^3} r^4 \hat{r}$

$$\Rightarrow \phi_{in}(r) \; (r \le a) = -\int_\infty^a \vec{E}_{out}\cdot d\vec{l} - \int_a^r \vec{E}_{in}\cdot d\vec{l}$$

only need ϕ inside sphere

$$= -\int_\infty^a \frac{\rho_0 a^3\, dr}{6\varepsilon_0 r^2} - \int_a^r \frac{\rho_0}{6\varepsilon_0 a^3} r^4\, dr$$

$$= -\frac{\rho_0}{6\varepsilon_0}\Big[\underbrace{a^3 \underbrace{\int_\infty^a \frac{dr}{r^2}}_{-\frac{1}{a}} + \frac{1}{a^3}\underbrace{\int_a^r r^4\, dr}_{\frac{1}{5}(r^5 - a^5)}}_{\left(-a^2 + \frac{1}{5a^3}r^5 - \frac{1}{5}a^2\right)}\Big]$$

$$\left(-a^2 + \frac{1}{5a^3}r^5 - \frac{1}{5}a^2\right) = \frac{1}{5a^3}r^5 - \frac{6}{5}a^2 = \frac{1}{5a^3}(r^5 - 6a^5)$$

$$\Rightarrow \phi_{in}(r) = \frac{\rho_0}{30\varepsilon_0 a^3}(6a^5 - r^5).$$

Now $\rho = 0$ for $r > a$

$$U = \frac{1}{2}\int \rho\phi\, dV = \frac{1}{2}\cdot\left(\frac{\rho_0}{30\varepsilon_0 a^3}\right)\left[\int_0^a \left(\frac{\rho_0}{a^3}r^3\right)(6a^5 - r^5) r^2\, dr\right]\underbrace{\int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta}_{4\pi}$$

$$= \frac{4\pi}{2}\frac{\rho_0}{30\varepsilon_0 a^3}\frac{\rho_0}{a^3}\int_0^a \Big(6a^5 \underbrace{r^5}_{\frac{1}{6}a^6} - \underbrace{r^{10}}_{\frac{1}{11}a^{11}}\Big)\, dr$$

$$= \frac{4\pi}{2}\frac{\rho_0^2}{30\varepsilon_0 a^6}\Big[\underbrace{6a^5 \cdot \frac{1}{6}a^6 - \frac{1}{11}a^{11}}_{\frac{10}{11}a^{11}}\Big]$$

$$\boxed{U = \frac{2\cancel{4}\pi}{\cancel{2}}\frac{1}{\cancel{30}_3}\frac{\cancel{10}^1}{11}\frac{\rho_0^2}{\varepsilon_0 a^6}a^{11} = \frac{2\pi}{33}\frac{\rho_0^2}{\varepsilon_0}a^5}$$ as it should.