Name: [redacted]

Student ID number: [redacted]

Oral interview time (see below): [redacted]

Discussion section: [redacted]

By signing [redacted]ou have neither given nor received unauthorized help on thi[redacted]

Signature: [redacted]

[redacted]IDTERM 2
PHYSICS 1B, SPRING 2020

MAY 22, 2020

READ THE FOLLOWING CAREFULLY:

- ▷ Open book. Calculators will not be needed. Make sure that you work independently on the exam, i.e., no discussion with others.
- ▷ Oral interview time: choose a time period of 20 minutes between 11:59AM and 11:59PM on May 23 (Pacific Time) for possible oral interview (via the zoom office hour link). A random sample of students will be selected and be notified via email by 11:59AM on May 23 (Pacific Time). If all of the work on your submission is your own, you will have nothing to worry about.
- ▷ For non-response in the field of oral interview time, I may assess a score penalty.
- ▷ Make sure to submit your exam packet via GradeScope by 9AM May 23, Pacific Time. **No credit will be given for late submissions.**
- ▷ **You must make sure that your submission has exactly the same number of pages as the posted exam PDF (including this page).**
- ▷ **You must justify your answers to each question.** Simply giving the correct answer without proper justification (can be brief) will not result in full credit.
- ▷ Mistakes in grading: If you find a mistake in the grading of your exam, alert the instructor within one week of the exams being returned.

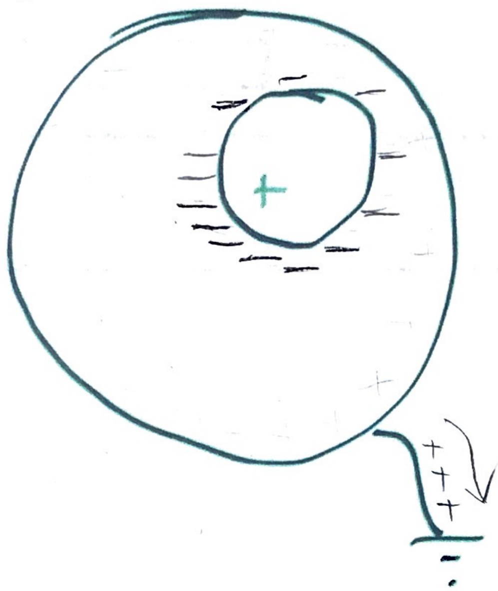
[1.] **Short answer conceptual questions**. Provide *concise* answers to the questions below; you should write enough to explain your answer, but an essay is not required (most can be answered in 2-3 sentences).

(a) (**10 pts**) During one of the in-class video demos, a kid stood on an insulator and placed her hands on the metal sphere of the Van der Graaf generator. This resulted in her hair standing on end (much like the below photo). Explain why this happened. If the kid wasn't standing on the insulator, would the same thing have happened? Why or why not?

The kid's hair began to stand up because electrons began to flow into her from the Van de Graaf generator, and when charges build up, they repel, pushing the hair strands apart.

This would not have happened if the kid were not standing on the insulator because the electrons would have left her into the ground. Thus, there would be no build up of charges (like sign).

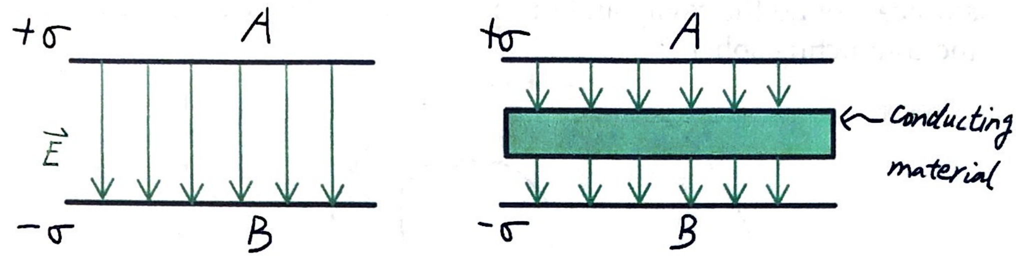
(b) **(10 pts)** A solid conducting sphere has an internal spherical cavity which is offset from the center of the sphere. Inside the cavity, but offset from the center of the cavity, I place a positive point charge $+q$. The conducting sphere is connected to ground via a thin wire as shown. Describe (either by drawing or words) any charges that exist on the surfaces of the conductor; the interior surface around the cavity and the external surface. What is the total charge on the conducting sphere?



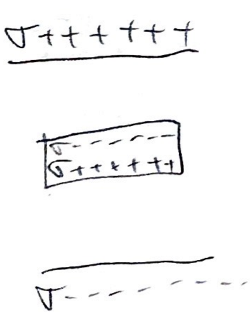
neutralization of positive charge by "outward flow of positive charge" or realistically, inward flow of electrons

There will be a negative charge distribution around the inner shell of the sphere, that is more concentrated near the positive charge, as indicated in the diagram. This is because the positive charge is closer to that area, so it will more strongly attract negative charges. Normally, because the negative charges flow towards the inner shell, there will be a positive charge distribution around the outer shell, but the sphere is grounded, so the remaining positive charges will be neutralized. Thus, the sphere has a total charge of $-q$.

(c) **(10 pts)** I have a capacitor consisting of two large parallel plates A and B, one charged positively (surface charge density $+\sigma$) and the other charged negatively (surface charge density $-\sigma$). If I insert a slab of conducting material as shown below, how does the absolute value of the potential difference between the two parallel plates A and B change? How does the capacitance change?



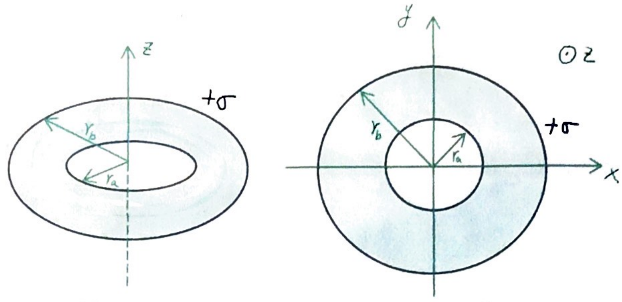
When a conducting slab is inserted, because there is no electric field within a conductor, the capacitor now can be seen as two parallel plate capacitors, in series, each with $C = \varepsilon_0 \frac{A}{D}$, except D is now different and smaller. Thus, each capacitor now has a capacitance greater than the original capacitor, and the total capacitance is also larger. The charge on each plate does not change, however, so if capacitance increases, potential difference between the plates of A and B will decrease. ($Q = C\Delta V$)



$Q = C\Delta V$

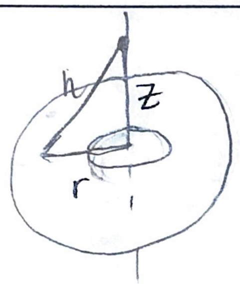
$Q = C\Delta V$

[2.] (30 pts) An annular ring with a uniform surface charge density $+\sigma$ sits in the xy plane, with its center at the origin of the coordinate. The annulus has an inner radius r_a and an outer radius r_b as shown below.



Side view   Top view

(a) What is the electric potential along the z-axis? Define the electric potential to be zero at infinity.



$$V(z) = \int_0^{2\pi}\int_{r_a}^{r_b} k\cdot\frac{\sigma}{h}\, r\cdot dr\cdot d\theta$$

$$V(z) = \left(\int_0^{2\pi} d\theta\right)\left(k\cdot\sigma\int_{r_a}^{r_b}\frac{r}{\sqrt{r^2+z^2}}\,dr\right)$$

$$V(z) = (2\pi)\,k\,\sigma\cdot\left((r^2+z^2)^{1/2}\Big|_{r_a}^{r_b}\right)$$

$$\boxed{V(z) = 2\pi k\sigma\cdot\left[(r_b^2+z^2)^{1/2} - (r_a^2+z^2)^{1/2}\right]}$$

assuming $V(\infty) = 0$

(b) From part (a), what is the electric potential in the limit $|z| \gg r_a$, $|z| \gg r_b$? What is the electric potential in the limit $|z| \ll r_a$, $|z| \ll r_b$? Hint: use the binomial expansion $(1+x)^\alpha \approx 1 + \alpha x$ for $|x| \ll 1$.

$$V(z) = 2\pi k\sigma \cdot \left[(r_b^2 + z^2)^{1/2} - (r_a^2 + z^2)^{1/2} \right]$$

$|z| >> r_a, |z| >> r_b$

$$V(z) = 2\pi k\sigma \cdot \left[\left(z^2\left(\frac{r_b^2}{z^2}+1\right)\right)^{1/2} - \left(z^2\left(\frac{r_a^2}{z^2}+1\right)\right)^{1/2} \right]$$

$$V(z) \approx 2\pi k\sigma \cdot \left[z\left[\left(1+\frac{1}{2}\left(\frac{r_b^2}{z^2}\right)\right) - \left(1+\frac{1}{2}\left(\frac{r_a^2}{z^2}\right)\right)\right] \right]$$

$$V(z) \approx 2\pi k\sigma z \left[\frac{1}{2z^2}(r_b^2 - r_a^2) \right]$$

$$\boxed{V(z) \approx \frac{\pi k\sigma}{z}(r_b^2 - r_a^2)}$$

$|z| << r_a, |z| << r_b$

$$V(z) = 2\pi k\sigma \left[\left(r_b^2\left(1+\frac{z^2}{r_b^2}\right)\right)^{1/2} - \left(r_a^2\left(1+\frac{z^2}{r_a^2}\right)\right)^{1/2} \right]$$

$$V(z) = 2\pi k\sigma \left[r_b \cdot (1+z^2/r_b^2)^{1/2} - r_a (1+z^2/r_a^2)^{1/2} \right]$$

$$V(z) \approx 2\pi k\sigma \left[r_b \cdot \left(1+\frac{1}{2}\left(\frac{z^2}{r_b^2}\right)\right) - r_a \cdot \left(1+\frac{1}{2}\left(\frac{z^2}{r_a^2}\right)\right) \right]$$

$$V(z) \approx 2\pi k\sigma \left[r_b + \frac{z^2}{2r_b} - r_a - \frac{z^2}{2r_a} \right]$$

$$\boxed{V(z) \approx 2\pi k\sigma \left[r_b - r_a + \frac{z^2}{2}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) \right]}$$

(c) A particle of mass m and negative charge $-q$ is constrained to move along the z-axis. If I kick the particle with a small displacement about the point $z = 0$, the particle will execute simple harmonic motion. Find the particle's frequency of oscillation. Hint: use the electric potential in the limit $|z| \ll r_a$, $|z| \ll r_b$ from part (b) and $E_z = -\partial V/\partial z$, where V is the electric potential.

$|z| << r_a, \ |z| << r_b$

$$V(z) = 2\pi k\sigma \left[r_b - r_a + \frac{z^2}{2}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)\right]$$

$$V(z) = 2\pi k\sigma (r_b - r_a) + 2\pi k\sigma \left(\frac{z^2}{2}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)\right)$$

$$E_z = -\frac{\partial V}{\partial z} = -\left(2\pi k\sigma z\left(\frac{1}{r_b} - \frac{1}{r_a}\right)\right)$$

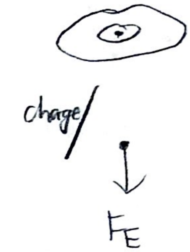
$$E_z = -2\pi k\sigma z\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

$\Sigma F_z = ma_z$

$F_E = ma_z$

$-q \cdot E_z = ma_z$

$-q \cdot \left(-2\pi k\sigma z\left(\frac{1}{r_b} - \frac{1}{r_a}\right)\right) = ma$



$$\frac{d^2z}{dt^2} - \frac{2\pi k\sigma z q}{m}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) = 0$$

$$w^2 = \frac{-2\pi k\sigma q}{m}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

$$\frac{d^2z}{dt^2} \underbrace{- \frac{2\pi k\sigma q}{m}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)}_{w^2} z = 0$$

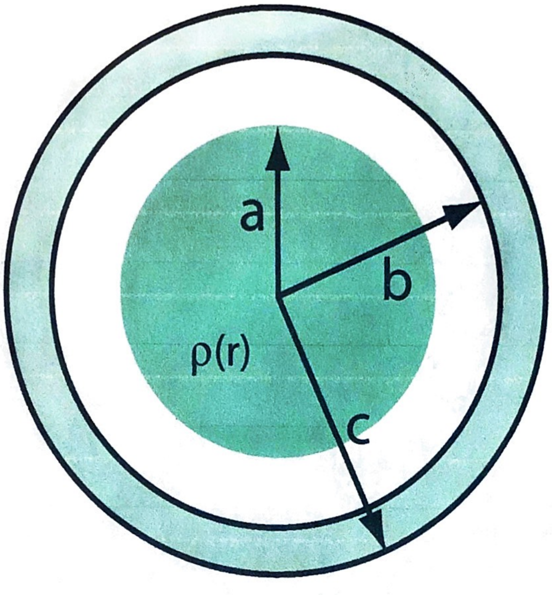
$$\boxed{w = \sqrt{\frac{2\pi k\sigma q}{m}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)} \text{ in rad/s}}$$

(sol. to diff Eq), prop. to displacement ✓

[3.] (**40 pts**) An insulating sphere of radius a has an embedded nonuniform charge density:

$$\rho(r) = \frac{\rho_o}{2}\left(1 + \frac{r}{a}\right)$$

where ρ_o is a positive constant (this charge density is only valid for $r \leq a$). Surrounding the sphere, and concentric with it, is a conducting shell of inner radius $b > a$ and outer radius c which is charged (not neutral). I do not know what the charge on the conductor is, but I am told that a measurement of the radial electric field right on the outer surface of the conducting shell (at $r = c + \epsilon$ where we take the limit $\epsilon \to 0$), shows that it is positive and equal to E_o.



$2\pi \rho_o \int_0^r r^2 + r^3/a \, dr$

$\frac{1}{3}r^3 + \frac{1}{4a}r^4 \Big|_0^{a}$

$2\pi \rho_o \left(\frac{1}{3}a^3 + \frac{1}{4}a^3\right)$

$2\pi \rho_o a^3 \cdot \frac{7}{12}$

(a) What is the total charge contained in the insulating sphere of radius a?

$$Q_{in} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\rho_o}{2}\left(1+\frac{r}{a}\right) \cdot r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\phi \, d\phi \cdot \frac{\rho_o}{2} \int_0^{a} r^2 + r^3/a \, dr$$

$$= 2\pi \cdot (2) \cdot \frac{\rho_o}{2}\left[\left(\frac{1}{3}r^3 + \frac{1}{4a}r^4\right)\Big|_0^a\right]$$

$$= 4\pi \cdot \frac{\rho_o}{2}\left[\frac{1}{3}a^3 + \frac{1}{4}a^3\right]$$

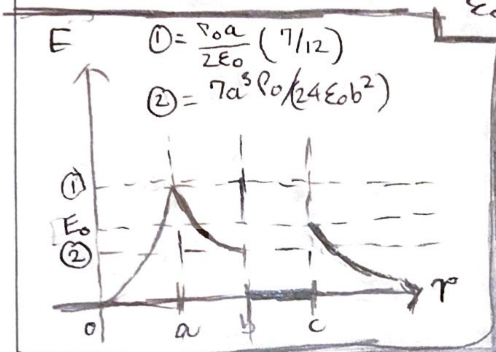
$$= 4\pi a^3 \cdot \frac{\rho_o}{2} \cdot \frac{7}{12}$$

$$\boxed{Q_{in} = \frac{7}{6}\pi a^3 \cdot \rho_o}$$

(b) Find the electric field everywhere in space: for $r < a$, $a < r < b$, $b < r < c$ and $r > c$. Plot the electric field versus radius.

$r<a$: $E(r)\cdot A = \dfrac{q_{inc}}{\varepsilon_0}$ (Gauss Law)

$$E(r)\cdot 4\pi r^2 = \frac{q_{inc}}{\varepsilon_0} = \frac{\int_0^r \frac{\rho_0}{2}\left(1+\frac{r}{a}\right)\cdot 4\pi r^2\,dr}{\varepsilon_0}$$



$$= \frac{2\pi\rho_0}{\varepsilon_0}\int_0^r r^2 + r^3/a\,dr$$

$$= \frac{2\pi\rho_0}{\varepsilon_0}\left(\frac{1}{3}r^3 + \frac{1}{4a}r^4\Big|_0^r\right)$$

$$= \frac{2\pi\rho_0}{\varepsilon_0}\left(\frac{1}{3}r^3 + \frac{1}{4a}r^4\right)$$

$$= \frac{2\pi\rho_0}{\varepsilon_0}\cdot r^3\left(\frac{1}{3} + \frac{1}{4a}r\right)$$

$$\boxed{E(r) = \frac{\rho_0 r}{2\varepsilon_0}\left(\frac{1}{3} + \frac{1}{4a}\cdot r\right),\ r<a}$$

$a<r<b$: $E(r)\cdot A = \dfrac{q_{inc}}{\varepsilon_0} = \dfrac{Q_{in}}{\varepsilon_0}$ (Gauss Law)

$$E(r)\cdot 4\pi r^2 = \frac{7\pi a^3\cdot\rho_0}{6\varepsilon_0}$$

$$\boxed{E(r) = \frac{7a^3\rho_0}{24\varepsilon_0 r^2},\ a<r<b}$$

$b<r<c$: E inside conductor is 0, so $\boxed{E(r) = 0,\ b<r<c}$

$r>c$: $d = c+\varepsilon$

$E(d) = E_0$

$$E_0\cdot 4\pi d^2 = \frac{q_{inc\,total}}{\varepsilon_0}$$

Gauss Law: $E(r)\cdot 4\pi r^2 = \dfrac{q_{inc\,total}}{\varepsilon_0} = E_0\cdot 4\pi d^2$

$$E(r) = E_0\cdot d^2/r^2$$

$d\to c$

$$\boxed{E(r) = E_0\cdot c^2/r^2,\ r>c}$$

(c) What is the total charge on the conducting shell in terms of the given information? What is the charge density on the inner and outer surfaces?

$$E(c) = E_0 = \frac{q_{inc\ total}}{\varepsilon_0 \cdot 4\pi c^2} \quad \text{(Gauss Law)}$$

$$q_{inc\ total} = E_0 \cdot \varepsilon_0 \cdot 4\pi c^2$$

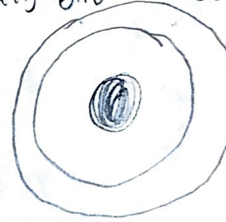
$$q_{shell} = q_{inc\ total} - q_{sphere}$$

$$\boxed{q_{shell} = E_0 \cdot \varepsilon_0 \cdot 4\pi c^2 - \frac{7}{6}\pi a^3 \cdot \rho_0}$$

$$\rho_{shell} = D$$

$$D_{inner} = \frac{q_{inner}}{4\pi b^2} \Rightarrow q_{inner} = -Q_{in}$$ (total charge of insulating sphere)

(draw a Gaussian surface inside shell, q_{in} must be 0, so $q_{inner} + Q_{in} = 0$)



$$D_{inner} = \frac{-\frac{7}{6}\pi a^3 \cdot \rho_0}{4\pi b^2}$$

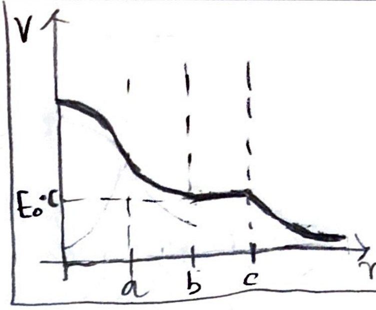
$$\boxed{D_{inner} = \frac{-7a^3\rho_0}{24b^2}}$$

$$D_{outer} = \frac{q_{outer}}{4\pi c^2} = \frac{q_{shell} - q_{inner}}{4\pi c^2} \quad (q_{inner} + q_{outer} = q_{shell})$$

$$D_{outer} = \frac{E_0 \cdot \varepsilon_0 \cdot 4\pi c^2 - \frac{7}{6}\pi a^3 \cdot \rho_0 + \frac{7}{6}\pi a^3 \rho_0}{4\pi c^2}$$

$$\boxed{D_{outer} = E_0 \cdot \varepsilon_0}$$

(d) Find the electric potential everywhere in space: for $r < a$, $a < r < b$, $b < r < c$ and $r > c$. Plot the electric potential versus radius. **Define the electric potential to be zero at infinity.**



$r > c$: $V(r) - V(\infty) = \int_r^{\infty} E(r)\,dr$

$V(r) = \int_r^{\infty} E_0 \cdot c^2/r^2\,dr$

$V(r) = E_0 \cdot c^2 \left[\left(-\frac{1}{r}\right)\Big|_r^{\infty}\right]$

$V(r) = E_0 \cdot c^2 \left(\frac{1}{r} - \frac{1}{\infty}\right)$

$$\boxed{V(r) = \frac{E_0 \cdot c^2}{r}, \quad r > c}$$

$b < r < c$: $V(r) - V(c) = \int_r^{c} E(r)\,dr$

$V(c) - V(\infty) = V(c) = E_0 \cdot c$

$V(r) - E_0 \cdot c = 0$

$$\boxed{V(r) = E_0 \cdot c, \quad b < r < c}$$

$a < r < b$: $V(r) - V(b) = \int_r^{b} E(r)\,dr$

$V(b) = V(r)$ when $b < r < c$ b/c $\Delta V = 0$ inside a conductor

$V(b) = E_0 \cdot c$

$V(r) = E_0 \cdot c + \int_r^{b} \frac{7a^3\rho_0}{24\varepsilon_0 r^2}\,dr$

$$\boxed{V(r) = E_0 \cdot c + \frac{7a^3\rho_0}{24\varepsilon_0} \cdot \left(-\frac{1}{b} + \frac{1}{r}\right), \quad a < r < b}$$

$r < a$: $V(r) - V(a) = \int_r^{a} E(r)\,dr$

$= \int_r^a \frac{\rho_0 \cdot r}{2\varepsilon_0}\left(\frac{1}{3} + \frac{1}{4a} r\right) dr = \frac{\rho_0}{2\varepsilon_0} \int_r^a \frac{r}{3} + \frac{r^2}{4a}\,dr$

$= \frac{\rho_0}{2\varepsilon_0}\left(\frac{1}{6} r^2 + \frac{1}{12a} r^3 \Big|_r^a\right)$

$= \frac{\rho_0}{2\varepsilon_0}\left(\frac{a^2}{6} + \frac{a^2}{12} - \frac{r^2}{6} - \frac{r^3}{12a}\right)$

$$\boxed{V(r) = \frac{\rho_0}{2\varepsilon_0}\left(\frac{a^2}{4} - \frac{r^2}{6} - \frac{r^3}{12a}\right) + E_0 \cdot c + \frac{7a^2\rho_0}{24\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), \quad r < a}$$

(e) What is the work that I do to assemble charges in this configuration, i.e., the total electric potential energy of this configuration? Use your answer to part (d), and the relationship between the electric potential energy and electric potential for a collection of charges.

$$PE_{sphere} = \frac{1}{2}\int_0^a \rho(r)\cdot 4\pi r^2 \cdot V(r)\,dr \quad , r<a$$

$$= \pi\rho_0 \int_0^a \left(1+\frac{r}{a}\right)\cdot r^2 \cdot \left(\frac{\rho_0}{2\varepsilon_0}\left(a^2/4 - r^2/6 - r^3/12a\right) + E_0\cdot C + \frac{7a^3\rho_0}{24\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)\right) dr$$

$$= \pi\rho_0 \int_0^a \left(r^2 + r^3/a\right)\left(\frac{\rho_0 a^2}{8\varepsilon_0} - \frac{\rho_0 r^2}{12\varepsilon_0} - \frac{\rho_0 r^3}{24a\varepsilon_0} + E_0\cdot C + \frac{7a^3\rho_0}{24\varepsilon_0 a} - \frac{7a^3\rho_0}{24\varepsilon_0 b}\right) dr$$

$$= \pi\rho_0 \int_0^a \frac{\rho_0 a^2 r^2}{8\varepsilon_0} - \frac{\rho_0 r^4}{12\varepsilon_0} - \frac{\rho_0 r^5}{24a\varepsilon_0} + E_0\cdot C\cdot r^2 + \frac{7a^2\rho_0 r^2}{24\varepsilon_0 a} - \frac{7a^3\rho_0 r^2}{24\varepsilon_0 b} + \frac{\rho_0 a r^3}{8\varepsilon_0} - \frac{\rho_0 r^5}{12\varepsilon_0 a} - \frac{\rho_0 r^6}{24a^2\varepsilon_0} + \frac{E_0\cdot C\cdot r^3}{a} + \frac{7a\rho_0 r^3}{24\varepsilon_0} - \frac{7a^2\rho_0 r^3}{24\varepsilon_0 b}\, dr$$

$$= \pi\rho_0\left[\frac{\rho_0 a^2 r^3}{24\varepsilon_0} - \frac{\rho_0 r^5}{60\varepsilon_0} - \frac{\rho_0 r^6}{144a\varepsilon_0} + \frac{E_0\cdot C r^3}{3} + \frac{7a^2\rho_0 r^3}{72\varepsilon_0} - \frac{7a^3\rho_0 r^3}{72\varepsilon_0 b} + \frac{\rho_0 a r^4}{32\varepsilon_0} - \frac{\rho_0 r^6}{72\varepsilon_0 a} - \frac{\rho_0 r^7}{168a^2\varepsilon_0} + \frac{E_0\cdot C r^4}{4a} + \frac{7a\rho_0 r^4}{96\varepsilon_0} - \frac{7a^2\rho_0 r^4}{96\varepsilon_0 b}\right]_0^a$$

$$= \pi\rho_0\left[\frac{\rho_0 a^5}{24\varepsilon_0} - \frac{\rho_0 a^5}{60\varepsilon_0} - \frac{\rho_0 a^5}{144\varepsilon_0} + \frac{E_0\cdot C\cdot a^3}{3} + \frac{7a^5\rho_0}{72\varepsilon_0} - \frac{7a^6\rho_0}{72\varepsilon_0 b} + \frac{\rho_0 a^5}{32\varepsilon_0} - \frac{\rho_0 a^5}{72\varepsilon_0} - \frac{\rho_0 a^5}{168\varepsilon_0} + \frac{E_0\cdot C a^3}{4} + \frac{7\rho_0 a^5}{96\varepsilon_0} - \frac{7a^6\rho_0}{96\varepsilon_0 b}\right]$$

$$= \pi\rho_0\left[\frac{420\rho_0 a^5}{10080\varepsilon_0} - \frac{168\rho_0 a^5}{10080\varepsilon_0} - \frac{70\rho_0 a^5}{10080\varepsilon_0} + \frac{E_0\cdot C\cdot a^3}{3} + \frac{980a^5\rho_0}{10080\varepsilon_0} - \frac{7a^6\rho_0}{72\varepsilon_0 b} + \frac{315\rho_0 a^5}{10080\varepsilon_0} - \frac{140\rho_0 a^5}{10080\varepsilon_0} - \frac{60\rho_0 a^5}{10080\varepsilon_0} + \frac{E_0\cdot C a^3}{4}\right.$$

Part e continued

$$+ \frac{735 P_0 a^5}{10080 \varepsilon_0} - \frac{7a^6 P_0}{96 \varepsilon_0 b}\Big]$$

$$= \pi P_0 \Big[\frac{2012 P_0 a^5}{10080 \varepsilon_0} + \frac{7E_0 \cdot C \cdot a^3}{12} - \frac{7a^6 P_0}{72 \varepsilon_0 b} - \frac{7a^6 P_0}{96 \varepsilon_0 b} \Big]$$

$$= \pi P_0 \Big[\frac{2012 P_0 a^5}{10080 \varepsilon_0} + \frac{7E_0 \cdot C \cdot a^3}{12} - \frac{28a^6 P_0}{288 \varepsilon_0 b} - \frac{21a^6 P_0}{288 \varepsilon_0 b} \Big]$$

$$\boxed{PE_{sphere} = \pi P_0 \Big[\frac{2012 P_0 a^5}{10080 \varepsilon_0} + \frac{7E_0 \cdot C \cdot a^3}{12} - \frac{49a^6 P_0}{288 \varepsilon_0 b} \Big]}$$

$PE_{shell} = (q_{inner} \cdot V_{inner} + q_{outer} \cdot V_{outer})/2$

$=$ $V_{inner} = V_{outer}$ ($\Delta V = 0$ inside conductor)

$= V\,(q_{inner} + q_{outer})/2$

$= V \cdot q_{shell}/2$

$$\boxed{PE_{shell} = \frac{E_0 \cdot C}{2} \cdot \left(E_0 \cdot \varepsilon_0 \cdot 4\pi c^2 - \frac{7}{6}\pi a^3 \cdot P_0\right)}$$

$$Work = PE_{sphere} + PE_{shell} = \pi P_0 \Big[\frac{503 P_0 a^5}{2520 \varepsilon_0} + \frac{7E_0 \cdot C \cdot a^3}{12} - \frac{49a^6 P_0}{288 \varepsilon_0 b} \Big] + \frac{E_0 \cdot C}{2}\left(E_0 \cdot \varepsilon_0 \cdot 4\pi c^2 - \frac{7}{6}\pi a^3 P_0\right)$$

$$Work = \frac{503\pi P_0^2 a^5}{1260 \varepsilon_0} + \cancel{\frac{7\pi E_0 C a^3 P_0}{12}} - \frac{49 a^6 P_0^2 \pi}{288 \varepsilon_0 b} + E_0^2 \varepsilon_0 2\pi c^3 - \cancel{\frac{7\pi a^3 P_0 E_0 C}{12}}$$

$$\boxed{Work = \frac{503\pi P_0^2 a^5}{2520 \varepsilon_0} - \frac{49 a^6 P_0^2 \pi}{288 \varepsilon_0 b} + E_0^2 \varepsilon_0 2\pi c^3}$$

(f) What is energy stored in the electric field? Use your answer to part (b), and the relationship between energy density and electric field.

Compare the electric field energy in this part with the total electric potential energy in part (e). Are they the same or not?

U_E = energy due to electric field

$$r<a: \; U_E = \int_0^a \frac{\varepsilon_0}{2}\left(\frac{\rho_0}{2\varepsilon_0}\left(\frac{r}{3}+\frac{r^2}{4a}\right)\right)^2 \cdot 4\pi r^2 dr$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\int_0^a r^2\left(\frac{r}{3}+\frac{r^2}{4a}\right)^2 dr$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\cdot\int_0^a r^2\left(\frac{r^2}{9}+\frac{r^3}{6a}+\frac{r^4}{16a^2}\right) dr$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\cdot\int_0^a \frac{r^4}{9}+\frac{r^5}{6a}+\frac{r^6}{16a^2}\, dr$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\cdot\left[\frac{1}{45}r^5+\frac{1}{36a}r^6+\frac{r^7}{112a^2}\right]\Bigg|_0^a$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\left[\frac{a^5}{45}+\frac{a^5}{36}+\frac{a^5}{112}\right]$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\left[\frac{112a^5+140a^5+45a^5}{5040}\right]$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\left[\frac{297a^5}{5040}\right] = \frac{\pi\rho_0^2}{2\varepsilon_0}\left[\frac{99a^5}{1680}\right]$$

$$= \frac{\pi\rho_0^2}{2\varepsilon_0}\cdot\frac{33a^5}{560}$$

$$U_E = \frac{\pi\rho_0^2\cdot 33a^5}{1120\varepsilon_0}, \; r<a$$

Part f continued

$a<r<b$: $U_E = \int_a^b \frac{\varepsilon_0}{2}\left(\frac{7a^3P_0}{24\varepsilon_0 r^2}\right)^2 \cdot 4\pi r^2 dr$

$U_E = 2\pi\varepsilon_0 \int_a^b r^2/r^4 \left(\frac{7a^3P_0}{24\varepsilon_0}\right)^2 dr$

$U_E = 2\pi\varepsilon_0 \cdot \left(\frac{7a^3P_0}{24\varepsilon_0}\right)^2 \cdot \left(\frac{-1}{r}\Big|_a^b\right)$

$U_E = 2\pi\varepsilon_0 \cdot \left(\frac{7a^3P_0}{24\varepsilon_0}\right)^2 \cdot \left(-\frac{1}{b}+\frac{1}{a}\right) = \underline{\frac{49\pi P_0^2 a^6}{288\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)}$

$b<r<c$: $U_E = 0 \quad (E=0)$

$r>c$: $U_E = \int_c^\infty \frac{\varepsilon_0}{2}\left(E_0 \cdot c^2/r^2\right)^2 \cdot 4\pi r^2 dr$

$U_E = 2\pi\varepsilon_0 E_0^2 \cdot \int_c^\infty \left(\frac{c^2}{r^2}\right)^2 \cdot r^2 dr$

$U_E = 2\pi\varepsilon_0 E_0^2 c^4 \cdot \int_c^\infty \frac{1}{r^2} dr$

$U_E = 2\pi\varepsilon_0 E_0^2 c^4 \cdot \left(-\frac{1}{r}\right)\Big|_c^\infty$

$\underline{U_E = 2\pi\varepsilon_0 E_0^2 c^3}$

$E_{TOTAL} = \frac{33\pi P_0^2 a^5}{1120\varepsilon_0} + \frac{49\pi P_0^2 a^6}{288\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) + 2\pi\varepsilon_0 E_0^2 c^3$

Once simplified, the energies are equal (electric field energy = electric potential energy), because energy stored in a system of electric charges is completely stored in the electric field. Thus, the 2 values should be equal.

$E_{TOTAL} = \frac{33\pi P_0^2 a^5}{1120\varepsilon_0} + \frac{49\pi P_0^2 a^5}{288\varepsilon_0} - \frac{49\pi P_0^2 a^6}{288\varepsilon_0 b} + 2\pi\varepsilon_0 E_0^2 c^3$

$E_{TOTAL} = \frac{(297+1715)\pi P_0^2 a^5}{10080\varepsilon_0} - \frac{49\pi P_0^2 a^6}{288\varepsilon_0 b} + 2\pi\varepsilon_0 E_0^2 c^3$

$E_{TOTAL} = \frac{503\pi P_0^2 a^5}{2520\varepsilon_0} - \frac{49\pi P_0^2 a^6 \pi}{288\varepsilon_0 b} + 2\pi\varepsilon_0 E_0^2 c^3$ = Work done in bringing charges together