Problem 1

30 points

In some diatomic molecules, we can model the force that one atom exerts on another as

$$F(r) = -\frac{A}{r^2} + \frac{B}{r^3},$$

where r is the separation of the molecules, and A and B are some constants.

(a): 10 points

Find the equilibrium separation r_0 of the atoms.

Equilibrium occurs when the net force is zero. Setting the force to zero, we find

$$\begin{aligned} 0 = F(r_0) &= -\frac{A}{r_0^2} + \frac{B}{r_0^3} \\ &= -Ar_0 + B, \end{aligned}$$

so that

$$r_0 = \frac{B}{A}.$$

We can then rewrite the force if we'd like, as

$$F(r) = A\left[-\frac{1}{r^2} + \frac{r_0}{r^3}\right]$$

(a): 10 points

(b): 10 points

Show that, if $r - r_0 \ll r_0$ (i.e. the atoms are close to their equilibrium separation), the motion of the atoms will be approximately simple harmonic.

We are to show that the force near $r = r_0$ is approximately linear. The first few derivatives of F are

$$\begin{aligned} \frac{df}{dr} &= A\left[\frac{2}{r^3} - \frac{3r_0}{r^4}\right] \\ \frac{d^2f}{dr^2} &= A\left[\frac{6}{r^4} - \frac{12r_0}{r^5}\right]. \end{aligned}$$

Thus the Taylor expansion of $F(r)$ around $r = r_0$ is

$$\begin{aligned} F(r) &= F(r_0) + A\left[\frac{2}{r_0^3} - \frac{3r_0}{r_0^4}\right](r - r_0) + \frac{1}{2}A\left[\frac{6}{r_0^4} - \frac{12r_0}{r_0^5}\right](r - r_0)^2 + \cdots \\ &= -\frac{A}{r_0^3}(r - r_0) - \frac{3A}{r_0^4}(r - r_0)^2 + \cdots. \end{aligned}$$

(b): 10 points

If $r - r_0 \ll r_0$, then the scond term here is much than the first, and we have

$$F(r) \approx -\frac{A}{r_0^3}(r - r_0).$$

The force is linear with respect to the displacement from equilibrium $r - r_0$, and so the motion is simple harmonic!

(b): 10 points

(c): 10 points

Determine the frequency of the simple harmonic motion.

The effective "spring constant" here is $\frac{A}{r_0^3} = \frac{A^4}{B^3}$. The frequency of simple harmonic motion with an effective spring constant of k is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{A^4}{mB^3}}.$$

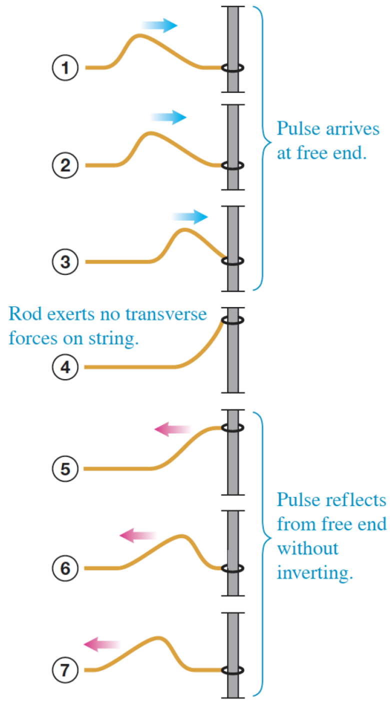
If the students found $f = \frac{1}{2\pi}\sqrt{\frac{A^4}{mB^3}}$, that's ok too.

(c): 10 points

Problem 2

40 points

Consider a rope of length L, tension T and mass per unit length μ that is held down (fixed) at the left end, and the right end is free to move, such as the one shown in this diagram:



A wave described by the wave function $y(x,t)$ with such a free end must satisfy $\frac{\partial y}{\partial x} = 0$ at the free end of the string. Find the allowed frequencies of the standing waves on this string in terms of μ, T, and/or L.

Let $x = 0$ be the fixed end of the string and $x = L$ be the free end. A general standing wave has the form

$$y(x,t) = (A\sin(kx) + B\cos(kx))\sin(\omega t + \phi).$$

Since $x = 0$ is fixed, $y(0,t) = 0$, and we need $B = 0$:

$$y(x,t) = A\sin(kx)\sin(\omega t + \phi).$$

Since $x = L$ is free, we must have $\frac{\partial y}{\partial x}|_L = Ak\cos(kx)\sin(\omega t + \phi) = 0$, which means we need $kL = (n + 1/2)\pi$ for integers n:

$$y(x,t) = A\sin\left((n + 1/2)\pi\frac{x}{L}\right)\sin(\omega t + \phi).$$

Frequency, wavenumber and wave speed are related by $\omega = vk$. The wave speed on this string is $v = \sqrt{T/\mu}$. Thus the allowed frequencies are

$$\omega_n = (n + 1/2)\pi\frac{1}{L}\sqrt{\frac{T}{\mu}}.$$

Problem 3

30 pts

(a): 10 pts

You visit the Liberty Bell on your trip to Philadelphia. While the guards aren't looking, you decide to test its oscillatory properties. You pull the bell back and let it go, noting that it takes 2 seconds to complete a full oscillation. You again grab the bell and pull it back 10% further. This time, it takes 2.2 seconds to complete a full oscillation. Can you describe the motion of the bell as simple harmonic? Explain your answer.

In simple harmonic motion, the period of motion is *independent* of the amplitude. Thus this cannot be described as simple harmonic motion.

(a): 10 pts

(b): 10 pts

Which, if any, of the following functions are valid pressure wave functions to describe a sound wave in open air (not necessarily in a tube)?

$$p_1(x,t) = p_0 e^{-(Ax-Bt)^2}$$
$$p_2(x,t) = p_0 e^{-Ax^2-Bt}$$
$$p_3(x,t) = p_0 \ln\left(\frac{x}{At}\right)$$

Here p_0, A, and B are constants. Justify your answer. If the answer is yes for any of them, solve for the speed of the wave in terms of p_0, A, and/or B.

For these to be valid waves, they must satisfy the wave equation. We can compute:

$$\begin{aligned}
\frac{\partial^2 p_1}{\partial x^2} &= \frac{\partial}{\partial x}\left(-2p_0 A(Ax-Bt)e^{-(Ax-Bt)^2}\right) \\
&= 4p_0 A^2(Ax-Bt)^2 e^{-(Ax-Bt)^2} - 2p_0 A^2 e^{-(Ax-Bt)^2} \\
\frac{\partial^2 p_1}{\partial t^2} &= \frac{\partial}{\partial t}\left(2p_0 B(Ax-Bt)e^{-(Ax-Bt)^2}\right) \\
&= 4p_0 B^2(Ax-Bt)^2 e^{-(Ax-Bt)^2} - 2p_0 B^2 e^{-(Ax-Bt)^2}.
\end{aligned}$$

We can see that $\frac{\partial^2 p_1}{\partial x^2} = -\frac{A^2}{B^2}\frac{\partial^2 p_1}{\partial t^2}$. Thus p_1 satisfies the wave equation, and is a valid wave. The speed can be found from the wave equation, since $\frac{\partial^2 p_1}{\partial x^2} = -\frac{1}{v^2}\frac{\partial^2 p_1}{\partial t^2}$. Thus the speed of p_1 is $v_1 = B/A$. Next,

(b): 10 pts

$$\begin{aligned}
\frac{\partial^2 p_2}{\partial x^2} &= \frac{\partial}{\partial x}\left(2p_0 Ax e^{-Ax^2-Bt}\right) \\
&= 4p_0 A^2 x^2 e^{-Ax^2-Bt} + 2p_0 A e^{-Ax^2-Bt} \\
\frac{\partial^2 p_2}{\partial t^2} &= \frac{\partial}{\partial t}\left(p_0 B e^{-Ax^2-Bt}\right) \\
&= p_0 B^2 e^{-Ax^2-Bt}.
\end{aligned}$$

For p_2, the second spatial and time derivatives are not proportional, and so p_2 does not satisfy the wave equation. p_2 does not describe a valid wave.

For p_3 is helpful to note that

$$p_3(x,t) = p_0[\ln(x) - \ln(t) - \ln(A)].$$

Then we can compute

$$\frac{\partial^2 p_3}{\partial x^2} = -\frac{p_0}{x^2}$$
$$\frac{\partial^2 p_3}{\partial t^2} = \frac{p_0}{t^2}.$$

These are not proportional, and so p_3 doesn't satisfy the wave equation, and isn't a valid wave.

(b): 10 pts

(c): 10 pts

For my wedding this summer, I opted for a subwoofer (speaker) that emits sound at a power of 20 W. The DJ warned me not to sit my grandma near the subwoofer. The threshold of painful sound is 120 dB. Assuming that the speaker emits sound in a spherically symmetric manner, how far does grandma need to sit away from the subwoofer so that she isn't in pain?

Assuming spherical symmetry, the intensity at a distance r is given by

$$I(r) = \frac{P}{4\pi r^2}.$$

The pain threshold is 120 dB, or

$$\beta_{\text{pain}} = 120 = 10\log_{10}(I_{\text{pain}}/I_0).$$

Solving for I_{pain}, we have

$$I_{\text{pain}} = I_0 10^{12} = 1.$$

Thus the distance she should sit away is

$$\begin{aligned} r &= \sqrt{\frac{P}{4\pi I_{\text{pain}}}} \\ &= \sqrt{\frac{P}{4\pi}} \\ &= \sqrt{\frac{20}{4\pi}} \\ &= 1.26 \text{ m}. \end{aligned}$$

(c): 10 pts