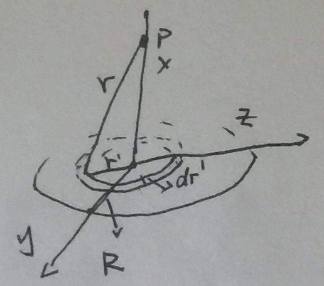
4. A disk with radius R has uniform surface charge density σ.

   (a) (10 points) By regarding the disk as series of thin concentric rings, calculate the electric potential V at a point on the disk's axis a distance x from the center of the disk. Assume that the potential is zero at infinity.

   (b) (10 points) Calculate the electric field using $\vec{E} = -\nabla V$.

(a) $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$, $dq = \sigma\, 2\pi r' dr'$, $r = \sqrt{r'^2 + x^2}$



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi r' dr'}{\sqrt{r'^2 + x^2}}$$

$$V = \int_0^R dV = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r' dr'}{\sqrt{r'^2 + x^2}}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r'^2 + x^2}\right]\Big|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2 + x^2} - x\right]$$

(b) $\vec{E} = -\nabla V$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial x}\left(\sqrt{R^2+x^2} - x\right)$$

$$= -\frac{\sigma}{2\varepsilon_0}\left(\frac{x}{\sqrt{R^2+x^2}} - 1\right)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{x}{\sqrt{R^2+x^2}}\right)$$

$$E_y = -\frac{\partial V}{\partial y} = 0$$

$$E_z = -\frac{\partial V}{\partial z} = 0$$

3. (20 points) A student enters the room to find an insulating sphere of radius a has an embedded nonuniform charge density:
$\rho(r) = \frac{\rho_0}{2}(1 + r/a)$, $r \leq a$,
where ρ_0 is positive constant charge density inside $r \leq a$. What is the electric field everywhere is space: considering the following regimes: $r < a$, $r > a$

① $r < a$

$$E_r(r<a)\ A = \frac{Q_{Enc}}{\varepsilon_0}$$

$$Q_{Enc} = \int_0^r dQ_{Enc} = \int_0^r \rho(r')\, 4\pi r'^2\, dr'$$

$$= \int_0^r \frac{\rho_0}{2}(1+r'/a)\, 4\pi r'^2\, dr'$$

$$= 2\pi\rho_0\left(\frac{r^3}{3} + \frac{r^4}{4a}\right)$$

$$E(r)(r<a)\ 4\pi r^2 = \frac{2\pi\rho_0\left(\frac{r^3}{3} + \frac{r^4}{4a}\right)}{\varepsilon_0}$$

$$E(r)(r<a) = \frac{\rho_0}{2\varepsilon_0}\left(\frac{r}{3} + \frac{r^2}{4a}\right)$$

② $r > a$

$$E_r(r>a)\ A = \frac{Q_{Enc}}{\varepsilon_0}$$

$$Q_{Enc} = \int_0^R dQ_{Enc} = \int_0^R \rho(r')\, 4\pi r'^2\, dr'$$

$$= \int_0^R \frac{\rho_0}{2}(1+r'/a)\, 4\pi r'^2\, dr' = \frac{7\pi\rho_0 a^3}{6}$$

$$E_r(r>a)\ 4\pi r^2 = \frac{7\pi\rho_0 a^3}{6\,\varepsilon_0}$$

$$E_r(r>a) = \frac{7\rho_0 a^3}{24\,\varepsilon_0 r^2}$$

2. (a) (15 points) Two thin rods of length L lie along the x-axis, one between x=a and x=a+L and the other between x=$-a$ and x=$-a-L$. Each rod has positive charge Q distributed uniformly along its length. Calculate the electric field produced by the second rod at points along the positive x-axis.

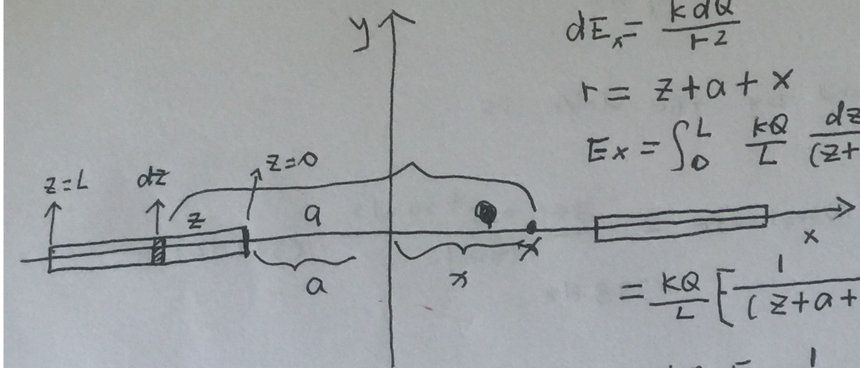
   (b) (10 points) A ring of radius "a" has a uniform charge per unit length and a total positive charge Q. Calculate the electric field at a point P along the axis of the ring at a distance x_0 from its center.

(a) $dQ = \frac{Q}{L}\,dz$

$dE_x = \frac{k\,dQ}{r^2}$

$r = z + a + x$

$E_x = \int_0^L \frac{kQ}{L}\frac{dz}{(z+a+x)^2}$

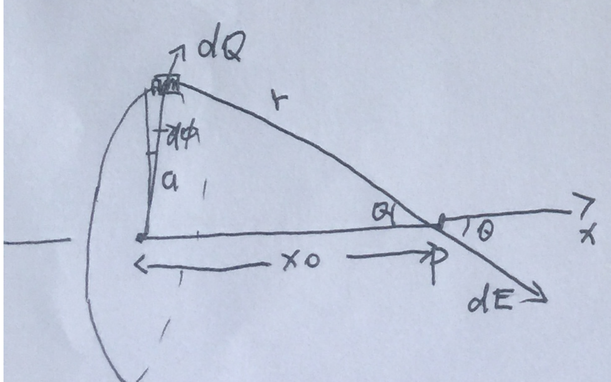


$= \frac{kQ}{L}\left[\frac{1}{(z+a+x)}\right]\Big|_0^L$

$= \frac{kQ}{L}\left[-\frac{1}{L+a+x} + \frac{1}{a+x}\right]$

$= \frac{kQ}{L}\left(\frac{1}{a+x} - \frac{1}{L+a+x}\right)$

(b)



$dE = k\frac{dQ}{r^2}$

$dE_x = k\frac{dQ}{r^2}\cos\theta$

$r = \sqrt{a^2 + x_0^2}$ $\quad \cos\theta = \frac{x_0}{r}$

$E_x = \int dE_x = \int\left(k\frac{dQ}{r^2}\right)\frac{x_0}{r} = k\frac{x_0}{r^3}\int dQ$

$= k\frac{x_0}{r^3}Q = \frac{k x_0 Q}{(a^2+x_0^2)^{3/2}}$

or $dQ = \lambda\, dl$ $\quad E_x = \int dE_x = \int k\frac{\lambda\, dl}{r^2}\frac{x_0}{r}$

$dl = a\, d\phi$ $\quad = \int \frac{k\lambda a\, d\phi\, x_0}{r^3} = \frac{k\lambda a x_0}{r^3}\int d\phi$

$\lambda = \frac{Q}{2\pi a}$ $\quad = \frac{k\lambda a x_0}{r^3}\cdot 2\pi = k\frac{Q a x^0}{2\pi a}\cdot 2\pi / r^3$

$= \frac{k x_0 Q}{(a_0^2 + x_0^2)^{\frac{3}{2}}}$

1. (15 points) A police car's siren emits a sinusoidal wave with frequency $f_s = 300\text{Hz}$. The speed of sound is 340m/s and the air is still. The police car is moving toward a warehouse at 30 m/s. What frequency does the driver hear reflected from the warehouse?

The frequency reaching the warehouse is,

$$f_{warehouse} = \frac{V}{V+V_S} f_S = \frac{340m/s}{340m/s+(-30m/s)} \cdot (300\,HZ)$$

$$= 329\,HZ$$

The frequency heard by the drive is.

$$f_L = \frac{V+V_L}{V} f_{warehouse} = \frac{340\,m/s + 30m/s}{340m/s} (329\,HZ)$$

$$= 358\,Hz.$$