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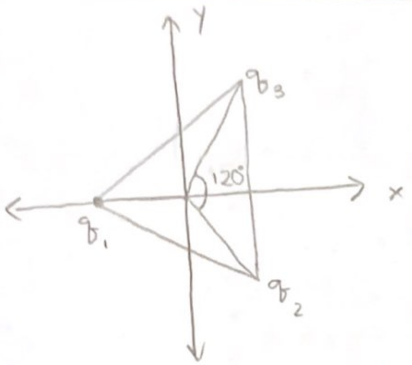
Instructions.

1. Complete this exam packet, and submit it to Gradescope before the end of the exam period.
2. Make sure all final work is in only the spaces provided, and make sure to upload a document of exactly the length of this packet.
3. If you are using a smartphone to generate a scan, please use a scanning app such as Adobe Scan to quickly generate an optimized PDF document.
4. There is extra space at the end of the packet in case the space below each problem isn't sufficient, but if you use that extra space, make sure to indicate that extra work is contained there when you show your work for a problem.
5. You are allowed to use the CCLE course page, the course textbook (OpenStax), and your notes, but you are not permitted to use any other internet resources.
6. A calculator (whatever type desired) is allowed.
7. You may not communicate about the contents of this exam with anyone during the exam period.
8. You may not logon to Campuswire during the exam period.
9. **Josh and the TAs will not be free to answer clarifying questions about the exam during the exam period.** If you believe there is an ambiguity, feel free to make a note of it to yourself and we will attempt to address it post-exam, but otherwise do your best to answer the questions based on what they say.
10. Violations of instructions pertaining to what you are not allowed to do during the exam will be considered cheating and will be reported to the Dean of Students.

1. **Treacherous Triangle Trickery**. Consider a charge distribution consisting of an equilateral triangle with a point charge q fixed at each of its vertices. Let d be the distance between the center of the triangle and each vertex, let the triangle's center be at the origin, and let one of its vertices lie on the x-axis at the point $x = -d$.

1.1. Compute the electric field at the center of the triangle by explicitly computing the sum of the electric fields due the charges at each vertex.



$\vec{r}_1 = \langle -d, 0, 0 \rangle$

$\vec{r}_2 = \langle d\cos(60), -d\sin(60), 0 \rangle = \langle \frac{d}{2}, -\frac{d\sqrt{3}}{2}, 0 \rangle$

$\vec{r}_3 = \langle d\cos(60), d\sin(60), 0 \rangle = \langle \frac{d}{2}, \frac{d\sqrt{3}}{2}, 0 \rangle$

$\vec{r} = \langle 0, 0, 0 \rangle$

$\vec{r} - \vec{r}_1 = \langle d, 0, 0 \rangle \qquad |\vec{r} - \vec{r}_1| = d$

$\vec{r} - \vec{r}_2 = \langle -\frac{d}{2}, \frac{d\sqrt{3}}{2}, 0 \rangle \qquad |\vec{r} - \vec{r}_2| = d$

$\vec{r} - \vec{r}_3 = \langle -\frac{d}{2}, -\frac{d\sqrt{3}}{2}, 0 \rangle \qquad |\vec{r} - \vec{r}_3| = d$

$$E(\vec{r}) = \sum_{i=1}^{N} \frac{k q_i}{|\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i)$$

$$E(\vec{r} = \langle 0,0,0 \rangle) = \frac{k q_1}{d^3}(\vec{r} - \vec{r}_1) + \frac{k q_2}{d^3}(\vec{r} - \vec{r}_2) + \frac{k q_3}{d^3}(\vec{r} - \vec{r}_3)$$

$q = q_1 = q_2 = q_3 \Rightarrow \frac{kq}{d^3}\left[\langle d,0,0 \rangle + \langle -\frac{d}{2}, -\frac{d\sqrt{3}}{2}, 0 \rangle + \langle -\frac{d}{2}, \frac{d\sqrt{3}}{2}, 0 \rangle \right]$

$$= \frac{kq}{d^3}\left[\langle 0,0,0 \rangle \right] = \boxed{0}$$

1.2. Let $V(x, y)$ be the electric potential as a function of position. Compute an expression for $V(x, y)$, and try to simplify it if possible.

$\vec{r} = \langle x, y \rangle \quad \vec{r}_1 = \langle -d, 0 \rangle \quad \vec{r}_2 = \langle \frac{d}{2}, -\frac{d\sqrt{3}}{2} \rangle \quad \vec{r}_3 = \langle \frac{d}{2}, \frac{d\sqrt{3}}{2} \rangle$

$$V(\vec{r}) = \sum_{i=1}^{N} \frac{k Q_i}{|\vec{r} - \vec{r}_i|}$$

$q = q_1 = q_2 = q_3$

$$V\langle x, y \rangle = \frac{k q_1}{\sqrt{(x+d)^2 + y^2}} + \frac{k q_2}{\sqrt{(x - \frac{d}{2})^2 + (y + \frac{d\sqrt{3}}{2})^2}} + \frac{k q_3}{\sqrt{(x - \frac{d}{2})^2 + (y - \frac{d\sqrt{3}}{2})^2}}$$

$$V(x, y) = kq \left(\frac{1}{\sqrt{(x+d)^2 + y^2}} + \frac{1}{\sqrt{(x - \frac{d}{2})^2 + (y + \frac{d\sqrt{3}}{2})^2}} + \frac{1}{\sqrt{(x - \frac{d}{2})^2 + (y - \frac{d\sqrt{3}}{2})^2}} \right)$$

1.3. If a point charge Q is placed at rest at the origin, will it remain at rest? Justify using electric potential and symbolic computation.

$$\nabla V(x,y) = U$$

$$\nabla V(x,y) = kq\left(\frac{(x+d)\hat{x} + y\hat{y}}{((x+d)^2+y^2)^{3/2}} + \frac{(x-\frac{d}{2})\hat{x} + (y+\frac{d\sqrt{3}}{2})\hat{y}}{((x-\frac{d}{2})^2 + (y+\frac{d\sqrt{3}}{2})^2)^{3/2}} + \frac{(x-\frac{d}{2})\hat{x} + (y-\frac{d\sqrt{3}}{2})\hat{y}}{((x-\frac{d}{2})^2 + (y-\frac{d\sqrt{3}}{2})^2)^{3/2}} \right)$$

$$\nabla V(0,0) = kq\left(\frac{d\hat{x}}{(d^2)^{3/2}} + \frac{(-\frac{d}{2})\hat{x} + (\frac{d\sqrt{3}}{2})\hat{y}}{((\frac{d^2}{4}) + \frac{d^2 3}{4})^{3/2}} + \frac{(-\frac{d}{2})\hat{x} + (-\frac{d\sqrt{3}}{2})\hat{y}}{((\frac{d^2}{4}) + \frac{d^2 3}{4})^{3/2}} \right)$$

$$= \frac{kq}{d^3}\left(d\hat{x} - \frac{d}{2}\hat{x} + \frac{d\sqrt{3}}{2}\hat{y} - \frac{d}{2}\hat{x} - \frac{d\sqrt{3}}{2}\hat{y} \right)$$

$$= \frac{kq}{d^3}(0) = 0$$

Because the electric field ~~energy~~ is ∅ at the origin,
a point charge Q will remain at rest at the origin
If a system of charges is in static equilibrium (at rest),
the electric field is 0

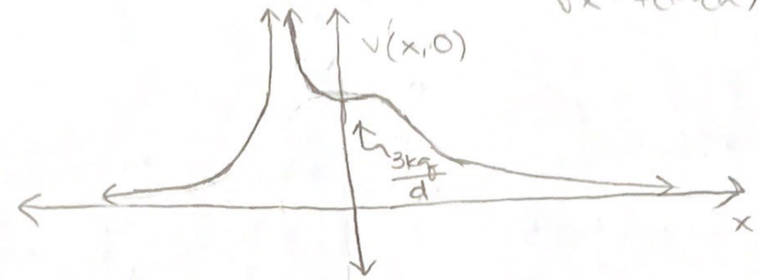
$$Eq = F$$

1.4. Sketch the graph of $V(x, 0)$ versus x.

$$v(x,0) = kq\left(\frac{1}{\sqrt{(x+d)^2}} + \frac{1}{\sqrt{(x-\frac{d}{2})^2+(\frac{d\sqrt{3}}{2})^2}} + \frac{1}{\sqrt{(x-\frac{d}{2})^2+(-\frac{d\sqrt{3}}{2})^2}}\right)$$

$$= kq\left(\frac{1}{\sqrt{x^2+2dx+d^2}} + \frac{1}{\sqrt{x^2+d^2-dx}} + \frac{1}{\sqrt{x^2+d^2-dx}}\right)$$

$$v(0,0) = \frac{kq3}{d}$$



1.5. Compute the Taylor expansion of $V(x, 0)$ about $x = 0$ up to the term of order x^2.

$$v(x,0) = kq\left(\frac{1}{(x^2+2dx+d^2)^{1/2}} + \frac{2}{(x^2+d^2-dx)^{1/2}}\right)$$

$$v'(x,0) = kq\left(-\frac{1}{\cancel{2}}(x^2+2dx+d^2)^{-3/2}(\cancel{2}x+\cancel{2}d) - \frac{\cancel{2}}{\cancel{2}}(x^2+d^2-dx)^{-3/2}(2x-d)\right)$$

$$v''(x,0) = kq\left(-\frac{1}{(x^2+2dx+d^2)^{3/2}} + \frac{3(2x+2d)^2}{4(x^2+2dx+d^2)^{5/2}} - \frac{2}{(x^2-dx+d^2)^{3/2}} + \frac{3(2x-d)^2}{2(x^2-dx+d^2)^{5/2}}\right)$$

$$v(0,0) = \frac{3kq}{d}$$

$$v'(0,0) = kq\left(-(d)(d^2)^{-3/2} - (-d)(d^2)^{-3/2}\right)$$

$$= kq\left(\frac{-d}{(d^2)^{3/2}} + \frac{d}{d^3}\right) = 0$$

$$v''(0,0) = kq\left(\frac{-1}{(d^2)^{3/2}} + \frac{3(2d)^2}{4(d^2)^{5/2}} - \frac{2}{(d^2)^{3/2}} + \frac{3(-d)^2}{2(d^2)^{5/2}}\right)$$

$$= kq\left(-\frac{1}{d^3} + \frac{6\cancel{d^2}}{4d^{\cancel{5}3}} - \frac{2}{d^3} + \frac{3\cancel{d^2}}{2d^{\cancel{5}3}}\right)$$

$$= \frac{3kq}{2d}$$

$$V(x,0) = \frac{3kq}{d} + \frac{3kq}{2d}\frac{x^2}{2!} = \frac{3kq}{d} + \frac{3kq}{4d}x^2$$

1.6. If a point charge Q is placed at the origin and then given a sufficiently small kick in the x-direction, will it remain in the vicinity fo the origin forever? Does it depend on the sign of Q? Does it matter if the kick is to the left or right? Justify all answers carefully.

$$\nabla V(x,0) = kq\left(\frac{(x+d)\hat{x}}{((x+d)^2)^{3/2}} + \frac{(x-\frac{d}{2})\hat{x} + \left(\frac{d\sqrt{3}}{2}\right)\hat{y}}{\left(\left(\frac{x-d}{2}\right)^2 + \left(\frac{d\sqrt{3}}{2}\right)^2\right)^{3/2}} + \frac{(x-\frac{d}{2})\hat{x} + \left(-\frac{d\sqrt{3}}{2}\right)\hat{y}}{\left(\left(\frac{x-d}{2}\right)^2 + \left(-\frac{d\sqrt{3}}{2}\right)^2\right)^{3/2}}\right)$$

$$\nabla V(x,0) = kq\left(\frac{(x+d)\hat{x}}{(x+d)^3} + \frac{(x-\frac{d}{2})\hat{x} + \frac{d\sqrt{3}}{2}\hat{y}}{\left(\frac{(x-d)^2}{4} + \frac{3d^2}{4}\right)^{3/2}} + \frac{(x-\frac{d}{2})\hat{x} - \frac{d\sqrt{3}}{2}\hat{y}}{\left(\frac{(x-d)^2}{4} + \frac{3d^2}{4}\right)^{3/2}}\right)$$

$$= -kq\left(\frac{(x+d)\hat{x}}{(x+d)^3} + \frac{2(x-\frac{d}{2})\hat{x}}{\left(\frac{(x-d)^2}{4} + \frac{3d^2}{4}\right)^{3/2}}\right)$$

notice that the gradient of the potential along the x-axis points along the x-axis, so it will not deviate from the x-axis

$\frac{d^2}{dx^2}V(x,0)\Big|_{x=0} = \frac{3}{2}\frac{kq}{d} > 0$, thus $V(x,0)$ is concave up at the origin

because the charge will stay in line with the x-axis and the concavity is positive at the origin, it will oscillate back and forth about the origin regardless of if it is kicked to the left or the right if it is positive, however, if it is negative, it will seek higher potential than the origin which is a local minimum. and will move towards $-d$ if kicked to the left and will oscillate about $\frac{d}{2}$ if kicked to the right which is a local maximum

1.7. If there is a case where the charge Q will oscillates under a small push in the x-direction, determine the period of small oscillations if the charge in the center has mass m. If there is not such a case of oscillatory motion, explain how you know this.

$F = ma \quad F = m\ddot{x}$

$Vq = U \quad -\nabla U = F(r)$

$U = kqQ$

$\frac{k}{m} = \omega = \sqrt{\frac{1}{2}}$

$\omega = \frac{1}{\sqrt{2}}$

$-\frac{3kqQ}{2d} = \frac{1}{2}\frac{3kqQ}{d}$

$t = 2\pi\sqrt{2}$

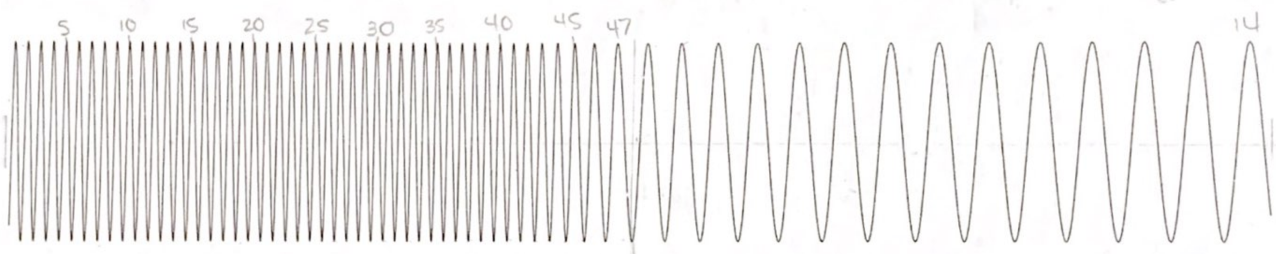
$\ddot{x} = -\frac{k}{m}x$ } harmonic oscillator eqn.

$= \boxed{2\sqrt{2}\pi}$

through the taylor expansion

$V''(x,0) = -\frac{1}{2}V(x,0)$

2. Auditory Airplane Inference. An airplane is flying past you some distance away at a constant speed in a straight line, and you use an app on your phone to record the sound it generates which has a constant emission frequency. The app outputs the following graph representing the pressure as a function of time in the air surrounding the phone. The graph displays 2 seconds of sensing data with the middle of the graph representing the moment of closest approach of the plane. Compute how fast the plane is going as accurately as you can, and clearly explain and show the logic and any algebra behind your computation.



$I = \frac{(\Delta P_{max})^2}{2\rho v} = \frac{2\pi^2 B f^2 s^2_{max}}{v} \qquad \cancel{2}\pi^2 B f^2 s^2_{max} = \frac{(\Delta P_{max})^2}{\cancel{2}\rho\, s_{max}^2}$

Intensity is proportional to the change in pressure

$f = \sqrt{\frac{(\Delta P_{max})^2}{\rho\, s_{max}^2\, \pi^2 B}} \qquad f = \Delta P_{max}\sqrt{\frac{1}{\rho\, s_{max}^2\, \pi B}}$

pressure variation

frequency observed changes with pressure variation perceived by the observed

↳ sound is the brain's response to pressure fluctuations in the air

According to the pressure graph, as the plane is approaching the observer, the frequency perceived is

$$f_o = 47\ Hz$$

As it is leaving, moving away from the observer, the frequency perceived is

$$f_o = 14\ Hz$$

$$f_o = \frac{v \pm v_o}{v \pm v_s} \qquad v_o = 0$$

$$47 = \frac{v}{v - v_s} f_s \qquad \frac{47(v - v_s)}{v} = 14\frac{(v + v_s)}{v}$$

$$14 = \frac{v}{v + v_s} f_s \qquad \frac{47}{14} = \frac{(v + v_s)}{(v - v_s)}$$

$$47v - 47v_s = 14v + 14v_s$$

$$33v = 61 v_s$$

$$v_s = \frac{33}{61} v \qquad v = \text{speed of sound} = 340\ m/s$$

$$v_s = \frac{33}{61} 340 = \boxed{183.93\ m/s}$$

3. True or False questions. Determine whether or not each of the following statements is true. If a statement is true, prove it. If the statement is false, provide a counterexample and explain how it constitutes a counterexample. Diagrams can be useful in explaining such things.

3.1. If the electric potential in a certain region of space is constant, then the charge enclosed by any closed surface completely contained within that region is zero.

$\boxed{\text{true}}$

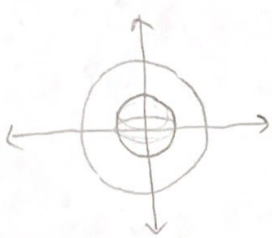
$$\Phi = \frac{q_{enc}}{\varepsilon_0} = \int \vec{E} \cdot \hat{n}\, dA$$

$$\vec{E} = -\nabla V$$

if V is constant, it's gradient is zero, thus the electric field is zero and if flux is dependent on the electric field, and if it is $\emptyset$, flux is also zero

$$\text{flux} = \Phi = \frac{q_{enc}}{\varepsilon_0} \qquad 0 = \frac{q_{enc}}{\varepsilon_0} \qquad q_{enc} = 0$$

3.2. A sphere of radius R is centered at the origin. A total charge Q is uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the the origin is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.



$\boxed{\text{true}}$

@ $r < \frac{R}{2}$ $\qquad q_{enc} = 0$

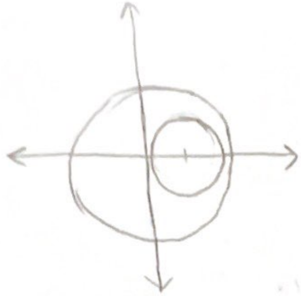
↑ radius

$$\Phi = \frac{0}{\varepsilon_0} = 0 = \int \vec{E} \cdot \hat{n}\, dA$$

due to rotational symmetry $\vec{E} = 0$, as electric field cancels

$\vec{E} = -\nabla V$, if the electric field inside the cavity is 0, V must be constant for its gradient to be zero

3.3. A sphere of radius R is centered at the origin. A total charge Q uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the point $(x, y, z) = (0, 0, R/4)$ is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.



$\boxed{\text{false}}$

$r<R$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{\cancel{3}Q}{\cancel{4\pi} R^3} \times \frac{\cancel{4}}{\cancel{3}}\cancel{\pi} r^3$$

$$4\pi \cancel{r^2}\, \vec{E} = \frac{Q r^{\cancel{3}}}{\varepsilon_0 R^3} \qquad \vec{E}_0(r) = \frac{Qr}{\varepsilon_0 R^3 4\pi}$$

in cavity

$$q_{enc} = \frac{\cancel{3}Q}{\cancel{4\pi} R^3} \times \frac{\cancel{4}}{\cancel{3}}\pi \left(\frac{R}{2} - r\right)^3$$

$$\oint E \hat{n}\, dA = \frac{Q}{\varepsilon_0 R^3}\left(R/2 - r\right)^3$$

$$E_1(r) 4\pi \cancel{\left(\frac{R}{2} - r\right)^2} = \frac{Q}{\varepsilon_0 R^3}\left(R/2 - r\right)^{\cancel{3}}$$

$$E_1(r) = \frac{Q\left(R/2 - r\right)}{\varepsilon_0 R^3 4\pi}$$

$$\vec{E}_0 - \vec{E}_1 = \vec{E}_{net} = \frac{Qr}{4\pi \varepsilon_0 R^3} - \frac{Q\left(R/2 - r\right)}{4\pi \varepsilon_0 R_3}$$

$$= \frac{Q\left(2r - R/2\right)}{4\pi \varepsilon_0 R^3} \neq 0$$

$\vec{E} \neq 0$ $\nabla V \neq 0$, therefore, electric potential is not constant inside the cavity