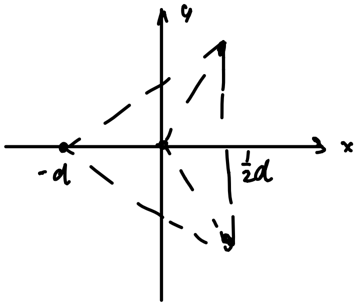
Instructions.

1. Complete this exam packet, and submit it to Gradescope before the end of the exam period.
2. Make sure all final work is in only the spaces provided, and make sure to upload a document of exactly the length of this packet.
3. If you are using a smartphone to generate a scan, please use a scanning app such as Adobe Scan to quickly generate an optimized PDF document.
4. There is extra space at the end of the packet in case the space below each problem isn't sufficient, but if you use that extra space, make sure to indicate that extra work is contained there when you show your work for a problem.
5. You are allowed to use the CCLE course page, the course textbook (OpenStax), and your notes, but you are not permitted to use any other internet resources.
6. A calculator (whatever type desired) is allowed.
7. You may not communicate about the contents of this exam with anyone during the exam period.
8. You may not logon to Campuswire during the exam period.
9. **Josh and the TAs will not be free to answer clarifying questions about the exam during the exam period**. If you believe there is an ambiguity, feel free to make a note of it to yourself and we will attempt to address it post-exam, but otherwise do your best to answer the questions based on what they say.
10. Violations of instructions pertaining to what you are not allowed to do during the exam will be considered cheating and will be reported to the Dean of Students.

1. Treacherous Triangle Trickery. Consider a charge distribution consisting of an equilateral triangle with a point charge q fixed at each of its vertices. Let d be the distance between the center of the triangle and each vertex, let the triangle's center be at the origin, and let one of its vertices lie on the x-axis at the point $x = -d$.

1.1. Compute the electric field at the center of the triangle by explicitly computing the sum of the electric fields due the charges at each vertex.



$$\vec{E} = \sum_{i=1}^{3} \frac{kq}{d^3}(\vec{r} - \vec{r_i})$$

$$= \frac{kq}{d^3}\left(d\hat{x} + \left(-\tfrac{1}{2}d\hat{x} - \tfrac{\sqrt{3}}{2}\hat{y}\right) + \left(-\tfrac{1}{2}d\hat{x} + \tfrac{\sqrt{3}}{2}\hat{y}\right)\right)$$

$$= \vec{0}$$

1.2. Let $V(x, y)$ be the electric potential as a function of position. Compute an expression for $V(x, y)$, and try to simplify it if possible.

$$V(x,y) = \sum_{i=1}^{3} \frac{kq}{|\vec{r} - \vec{r_i}|}$$

$\vec{r_1} = \langle -d, 0 \rangle$

$\vec{r_2} = \langle \tfrac{1}{2}d, \tfrac{\sqrt{3}}{2}d \rangle$

$\vec{r_3} = \langle \tfrac{1}{2}d, -\tfrac{\sqrt{3}}{2}d \rangle$

$$= kq \cdot \left(\frac{1}{\sqrt{(x+d)^2 + y^2}} + \frac{1}{\sqrt{(x-\frac{1}{2}d)^2 + (y - \frac{\sqrt{3}}{2}d)^2}} + \frac{1}{\sqrt{(x-\frac{1}{2}d)^2 + (y+\frac{\sqrt{3}}{2}d)^2}} \right)$$

$$= kq \cdot \left(\frac{1}{\sqrt{x^2+y^2+2dx+d^2}} + \frac{1}{\sqrt{x^2+y^2+d^2-dx-\sqrt{3}dy}} + \frac{1}{\sqrt{x^2+y^2+d^2-dx+\sqrt{3}dy}} \right)$$

1.3. If a point charge Q is placed at rest at the origin, will it remain at rest? Justify using electric potential and symbolic computation.

$$-\vec{\nabla}V(x,y) = \frac{1}{2}kq \cdot \left\langle \frac{2x+2d}{\left[(x+d)^2+y^2\right]^{\frac{3}{2}}} + \frac{2x-d}{\left[(x-\frac{d}{2})^2+(y-\frac{\sqrt{3}}{2}d)^2\right]^{\frac{3}{2}}} + \frac{2x-d}{\left[(x-\frac{d}{2})^2+(y+\frac{\sqrt{3}}{2}d)^2\right]^{\frac{3}{2}}}, \right.$$

$$\left. \frac{2y}{\left[(x+d)^2+y^2\right]^{\frac{3}{2}}} + \frac{2y-\sqrt{3}d}{\left[(x-\frac{d}{2})^2+(y-\frac{\sqrt{3}}{2}d)^2\right]^{\frac{3}{2}}} + \frac{2y+\sqrt{3}d}{\left[(x-\frac{d}{2})^2+(y+\frac{\sqrt{3}}{2}d)^2\right]^{\frac{3}{2}}} \right\rangle$$

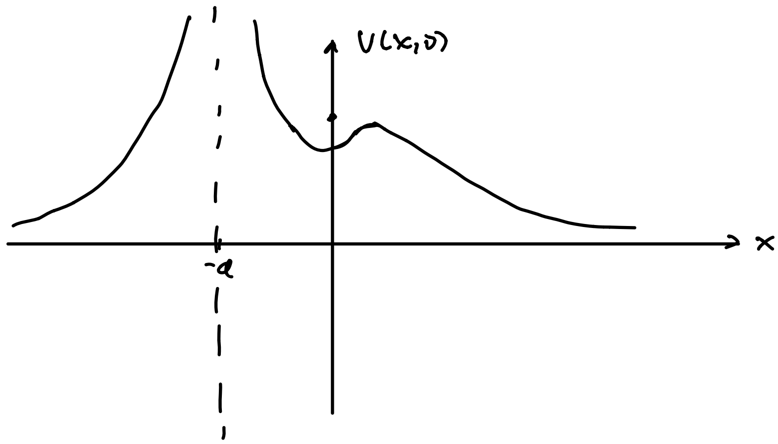
$$-\vec{\nabla}V(0,0) = \frac{1}{2}kq \left\langle \frac{2d}{d^3} + \frac{-d}{d^3} + \frac{-d}{d^3},\ 0 + \frac{-\sqrt{3}d}{d^3} + \frac{\sqrt{3}d}{d^3} \right\rangle$$

$$= \vec{0} = \vec{E}(0,0)$$

$\therefore$ There is zero net field at origin, and the charge would stay at rest.

1.4. Sketch the graph of $V(x, 0)$ versus x.

$$V(x,0) = kq\left(\frac{1}{|x+d|} + \frac{2}{\sqrt{x^2+d^2-dx}} \right)$$



1.5. Compute the Taylor expansion of $V(x, 0)$ about $x = 0$ up to the term of order x^2.

$$V(x,0) = kq\left(\frac{1}{|x+d|} + \frac{2}{\sqrt{x^2+d^2-dx}} \right)$$

$$V(0) = kq\left(\frac{1}{d} + \frac{2}{d} \right) = \frac{3kq}{d}$$

$$V'(x) = kq\left(-\frac{1}{(x+d)^2} - \frac{2x-d}{(x^2+d^2-dx)^{\frac{3}{2}}} \right) \qquad V'(0) = kq\left(-\frac{1}{d^2} - \frac{-d}{d^3} \right)$$

$$= 0$$

$$V''(x) = kq\left(\frac{2}{(x+d)^3} - \left(\frac{2(x^2+d^2-dx)^{\frac{3}{2}} - \frac{3}{2}(2x-d)^2(x^2+d^2-dx)^{\frac{1}{2}}}{(x^2+d^2-dx)^3} \right) \right)$$

$$V''(0) = kq\left(\frac{2}{d^3} - \left(\frac{2d^3 - \frac{3}{2}d^2 \cdot d}{d^6} \right) \right) = kq\left(\frac{2}{d^3} - \frac{1}{2} \cdot \frac{1}{d^3} \right) = \frac{3}{2}\frac{kq}{d^3}$$

$$\therefore V(x,0) \approx \frac{3kq}{d} + \frac{3}{4}\frac{kq}{d^3}x^2$$

1.6. If a point charge Q is placed at the origin and then given a sufficiently small kick in the x-direction, will it remain in the vicinity fo the origin forever? Does it depend on the sign of Q? Does it matter if the kick is to the left or right? Justify all answers carefully.

$\vec{\nabla} V(x,0) \approx \frac{3}{2}\frac{kq}{d^3} x\,\hat{x}$ according to the approximationof Taylor expansion.

$-\vec{\nabla} V(x,0) = \vec{E}(x,0)$, so in order to keep the charge in vicinity of origin, the force should have opposite direction to the displacement x for a small push. $\vec{F} = \vec{E}q \approx -\frac{3}{2}\frac{kqQ}{d^3} x\,\hat{x}$

Therefore, Q must have same sign as q, in order to experience force that is opposite to its displacement on x-axis. If $q>0$, $Q>0$, and if $q<0$ $Q<0$, otherwise it will not stay at origin.

It does not depend on whether left or right kick.

When kick to left, $x<0$, $Q \cdot q > 0$, $|\vec{F}| > 0$ along x-axis

When kick to right $x>0$, $Q \cdot q > 0$, $|\vec{F}| < 0$ along x-axis.

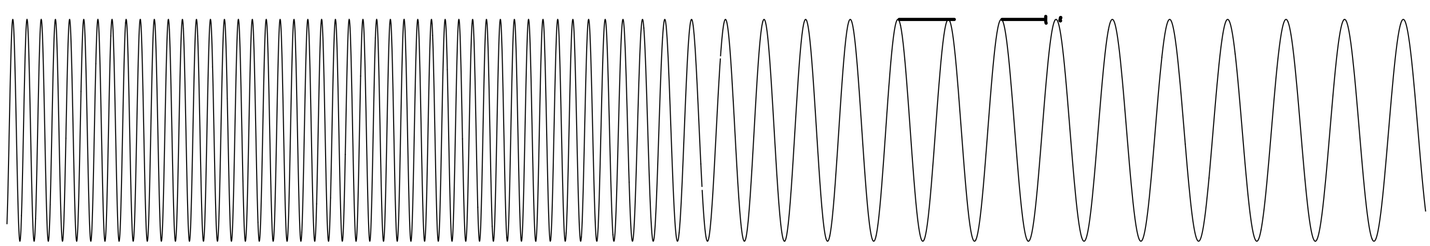
So no matter how the charge is kicked, it stay at origin.

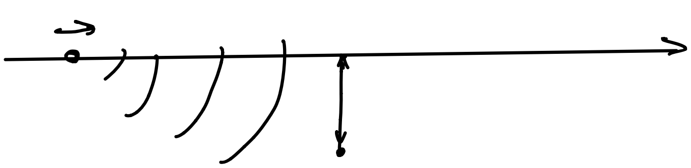
1.7. If there is a case where the charge Q will oscillates under a small push in the x-direction, determine the period of small oscillations if the charge in the center has mass m. If there is not such a case of oscillatory motion, explain how you know this.

$$|\vec{F}| = -\frac{3}{2}\frac{kqQ}{d^3}x = m\ddot{x} \qquad \therefore \omega^2 = \frac{3}{2}\frac{kqQ}{d^3 m}$$

$$\omega = \sqrt{\frac{3kqQ}{2d^3 m}} \qquad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{3kqQ}{2d^3 m}}} = \sqrt{\frac{8\pi^2 d^3 m}{3kqQ}} \qquad (Q>0)$$

2. Auditory Airplane Inference. An airplane is flying past you some distance away at a constant speed in a straight line, and you use an app on your phone to record the sound it generates which has a constant emission frequency. The app outputs the following graph representing the pressure as a function of time in the air surrounding the phone. The graph displays 2 seconds of sensing data with the middle of the graph representing the moment of closest approach of the plane. Compute how fast the plane is going as accurately as you can, and clearly explain and show the logic and any algebra behind your computation.





Assume emission frequency is f_s

When plane is far away from the observer, the frequency received is

$$f = f_s \frac{v}{v - v_s} = \frac{1}{T_1}$$ when plane is moving towards observer

and $f = f_s \frac{v}{v + v_s} = \frac{1}{T_2}$ when plane is away from the observer.

since it can be approximated as situation when observer and source are at same line.

By measuring using ruler, $T_1 = 2 \cdot \frac{0.3}{23} \approx 0.026\ s$

$T_2 = 2 \cdot \frac{1.1}{23} \approx 0.096\ s$, since overall length is 23 cm, and distances between wavefronts when moving ~~towards~~ and away are 0.3 cm and 1.1 cm. Take speed of sound as $343\ ms^{-1}$

$f_1 = \frac{1}{0.026} = f_s \frac{343}{343 - v_s}$ $\quad f_2 = \frac{1}{0.096} = f_s \frac{343}{343 + v_s}$

$\frac{0.096}{0.026} \approx 3.67 = \frac{343 + v_s}{343\ v_s}$ $\quad 3.67\ (343 + v_s) = 343 - v_s$

$1266.4 - 3.67\ v_s = 343 + v_s$

$v_s = 195\ ms^{-1} \approx 705\ km/h$

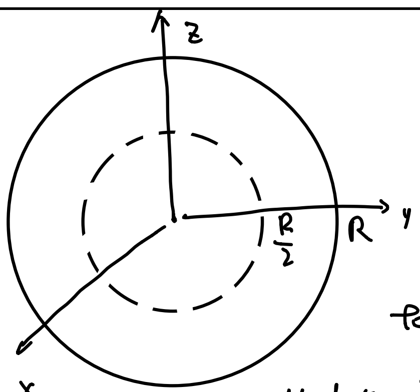
3. True or False questions. Determine whether or not each of the following statements is true. If a statement is true, prove it. If the statement is false, provide a counterexample and explain how it constitutes a counterexample. Diagrams can be useful in explaining such things.

3.1. If the electric potential in a certain region of space is constant, then the charge enclosed by any closed surface completely contained within that region is zero.

True. If the electric potential in the certain region is constant, then the negative gradient of potential is zero, which means the electric field exerted in the region is zero. According to Gauss's law,

$$\Phi = \int \vec{E} \cdot \hat{n}\, dA = \frac{q_{enc}}{\varepsilon_0} = 0 \qquad \therefore\ q_{enc} = 0$$

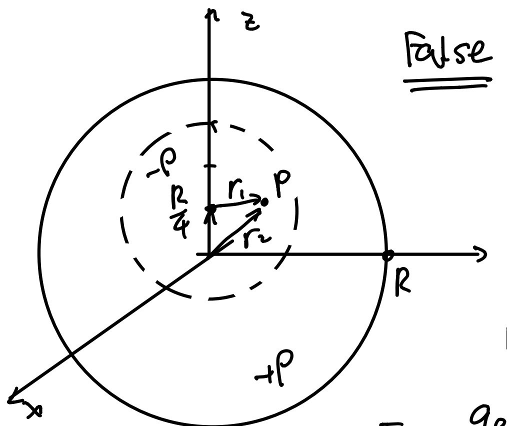
3.2. A sphere of radius R is centered at the origin. A total charge Q is uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the the origin is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.



True. Constructing a spherical Gaussian surface centered at origin and enclose the cavity, there is no charge in the cavity, and according to Gauss's law, $\frac{q_{enc}}{\varepsilon_0} = \Phi = 0$, thus $\vec{E}$ is either parallel to $\hat{n}$ or equals to zero. due to symmetry of the spherical charge distribution, $\vec{E}$ is pointing at direction through origin radically so it cannot be parallel to $\hat{n}$. thus $\vec{E} = 0$

Since $-\vec{\nabla} V = \vec{E}$, V must be constant everywhere in the cavity.

3.3. A sphere of radius R is centered at the origin. A total charge Q uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the point $(x, y, z) = (0, 0, R/4)$ is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.



False, suppose the charge density in the big sphere is ρ and total charge is Q $\quad \rho = \frac{Q}{\frac{4}{3}\pi R^3}$

According to Gauss's law, if there is no cavity inside the sphere, the total charge enclosed by sphere with radius r is $q_{enc} = \frac{4}{3}\rho\pi r^3$

$$\Phi = \frac{q_{enc}}{\varepsilon_0} = \frac{4\rho\pi r^3}{3\varepsilon_0} = \int \vec{E}\cdot\hat{n}\,dA = |\vec{E}|\,4\pi r^2$$

due to the symmetry of charge distribution and electric field through the sphere.

$$\therefore |\vec{E}| = \frac{\rho r}{3\varepsilon_0} \qquad \vec{E} = \frac{\rho\vec{r}}{3\varepsilon_0}$$

Since there is a cavity, we can assume that there are positive charges with density $+\rho$ in big sphere and $-\rho$ charges in the small cavity. In small cavity,

similarily, $\vec{E} = \frac{-\rho\vec{r}}{3\varepsilon_0}$

Let there be a point p in the cavity. respect to the center of cavity the position vector of p is $\vec{r}_1$, and with respect to the outer sphere the position vector of P is $\vec{r}_2$. Thus, the total electric field on p is

$$\vec{E} = \vec{E}_-(\vec{r}_1) + \vec{E}_+(\vec{r}_2) = \frac{-\rho\vec{r}_1}{3\varepsilon_0} + \frac{\rho\vec{r}_2}{3\varepsilon_0} = \frac{\rho}{3\varepsilon_0}(\vec{r}_2 - \vec{r}_1)$$

$\vec{r}_2 - \vec{r}_1$ is the vector pointing from $z=0$ to $z=\frac{R}{4}$

$$\therefore \vec{r}_2 - \vec{r}_1 = \frac{R}{4}\hat{z} \qquad \vec{E} = \frac{\rho R}{12\varepsilon_0}\hat{z} = \frac{Q}{16\pi R^2\varepsilon_0}\hat{z} \neq \vec{0}$$

which is a uniform upward field.

(continue to extra space)

Space for extra work.

Since $-\vec{\nabla} V = \vec{E}$, when electric field is not zero, potential cannot be constant anywhere. So the statement is not true.

$V = -\int \vec{E} \cdot d\vec{l}$

Furthermore, conceptually, The distribution now only has symmetry along z-axis, and the field must point along z-axis for point on the axis. At center of cavity, there are more charge at bottom and less charge at top. If $Q > 0$, field is upward with a magnitude. $\vec{E}$ would not be zero on the axis, thus potential cannot be constant.

Space for extra work.