Instructions.

1. Complete this exam packet, and submit it to Gradescope before the end of the exam period.
2. Make sure all final work is in only the spaces provided, and make sure to upload a document of exactly the length of this packet.
3. If you are using a smartphone to generate a scan, please use a scanning app such as Adobe Scan to quickly generate an optimized PDF document.
4. There is extra space at the end of the packet in case the space below each problem isn't sufficient, but if you use that extra space, make sure to indicate that extra work is contained there when you show your work for a problem.
5. You are allowed to use the CCLE course page, the course textbook (OpenStax), and your notes, but you are not permitted to use any other internet resources.
6. A calculator (whatever type desired) is allowed.
7. You may not communicate about the contents of this exam with anyone during the exam period.
8. You may not logon to Campuswire during the exam period.
9. **Josh and the TAs will not be free to answer clarifying questions about the exam during the exam period**. If you believe there is an ambiguity, feel free to make a note of it to yourself and we will attempt to address it post-exam, but otherwise do your best to answer the questions based on what they say.
10. Violations of instructions pertaining to what you are not allowed to do during the exam will be considered cheating and will be reported to the Dean of Students.

1. Treacherous Triangle Trickery. Consider a charge distribution consisting of an equilateral triangle with a point charge q fixed at each of its vertices. Let d be the distance between the center of the triangle and each vertex, let the triangle's center be at the origin, and let one of its vertices lie on the x-axis at the point $x = -d$.

1.1. Compute the electric field at the center of the triangle by explicitly computing the sum of the electric fields due the charges at each vertex.

1.2. Let $V(x, y)$ be the electric potential as a function of position. Compute an expression for $V(x, y)$, and try to simplify it if possible.

1.3. If a point charge Q is placed at rest at the origin, will it remain at rest? Justify using electric potential and symbolic computation.

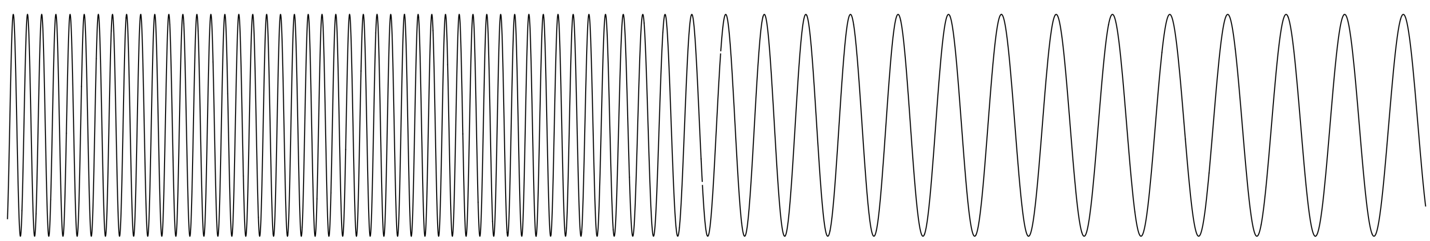
1.4. Sketch the graph of $V(x, 0)$ versus x.

1.5. Compute the Taylor expansion of $V(x, 0)$ about $x = 0$ up to the term of order x^2.

1.6. If a point charge Q is placed at the origin and then given a sufficiently small kick in the x-direction, will it remain in the vicinity fo the origin forever? Does it depend on the sign of Q? Does it matter if the kick is to the left or right? Justify all answers carefully.

1.7. If there is a case where the charge Q will oscillates under a small push in the x-direction, determine the period of small oscillations if the charge in the center has mass m. If there is not such a case of oscillatory motion, explain how you know this.

2. Auditory Airplane Inference. An airplane is flying past you some distance away at a constant speed in a straight line, and you use an app on your phone to record the sound it generates which has a constant emission frequency. The app outputs the following graph representing the pressure as a function of time in the air surrounding the phone. The graph displays 2 seconds of sensing data with the middle of the graph representing the moment of closest approach of the plane. Compute how fast the plane is going as accurately as you can, and clearly explain and show the logic and any algebra behind your computation.



3. **True or False questions.** Determine whether or not each of the following statements is true. If a statement is true, prove it. If the statement is false, provide a counterexample and explain how it constitutes a counterexample. Diagrams can be useful in explaining such things.

3.1. If the electric potential in a certain region of space is constant, then the charge enclosed by any closed surface completely contained within that region is zero.

3.2. A sphere of radius R is centered at the origin. A total charge Q is uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the the origin is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.

3.3. A sphere of radius R is centered at the origin. A total charge Q uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the point $(x, y, z) = (0, 0, R/4)$ is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.

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1. Treacherous Triangle Trickery. Consider a charge distribution consisting of an equilateral triangle with a point charge q fixed at each of its vertices. Let d be the distance between the center of the triangle and each vertex, let the triangle's center be at the origin, and let one of its vertices lie on the x-axis at the point $x = -d$.

1.1. Compute the electric field at the center of the triangle by explicitly computing the sum of the electric fields due the charges at each vertex.

This can be done by drawing a picture and finding components using plane geometry or just by using the expression for the electric field of a point charge[a] + symbolic computation. Let's try the latter way, noticing that the the contribution for each charge has a factor kq that can be factored out of the whole expression, and noting that since all charges are in the x-y plane, we can effectively treat this as a 2-dimensional problem and ignore the z-direction:

$$\mathbf{E}(0,0) = kq\left[\frac{(0-d/2)\hat{\mathbf{x}} + (0-d\sqrt{3}/2)\hat{\mathbf{y}}}{|(0-d/2)\hat{\mathbf{x}} + (0-d\sqrt{3}/2)\hat{\mathbf{y}}|^3} + \frac{(0-d/2)\hat{\mathbf{x}} + (0-(-d\sqrt{3}/2))\hat{\mathbf{y}}}{|(0-d/2)\hat{\mathbf{x}} + (0-(-d\sqrt{3}/2))\hat{\mathbf{y}}|^3} + \frac{(0-(-d))\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}}}{|(0-(-d))\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}}|^3}\right] \tag{1}$$

$$= \frac{kq}{d^3}\Big((d/2 - d/2 + d)\hat{\mathbf{x}} + (-d\sqrt{3}/2 + d\sqrt{3}/2)\hat{\mathbf{y}}\Big) \tag{2}$$

$$= \mathbf{0} \tag{3}$$

[a] The field at point $\mathbf{r}$ of point charge q at location $\mathbf{r}'$ is $kq(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^3$

1.2. Let $V(x, y)$ be the electric potential as a function of position. Compute an expression for $V(x, y)$, and try to simplify it if possible.

We use the expression for the electric potential due to a point charge[a] and add the potential due to all point charges together noting that the common factor kq can be factored out;

$$V(x,y) = kq\left[\frac{1}{|(x-d/2)\hat{\mathbf{x}} + (y-d\sqrt{3}/2)\hat{\mathbf{y}}|} + \frac{1}{|(x-d/2)\hat{\mathbf{x}} + (y-(-d\sqrt{3}/2))\hat{\mathbf{y}}|} + \frac{1}{|(x-d)\hat{\mathbf{x}} + (y-0)\hat{\mathbf{y}}|}\right] \tag{4}$$

$$= kq\left[\frac{1}{\sqrt{(x-d/2)^2 + (y-d\sqrt{3}/2)^2}} + \frac{1}{\sqrt{(x-d/2)^2 + (y+d\sqrt{3}/2)^2}} + \frac{1}{\sqrt{(x+d)^2 + y^2}}\right] \tag{5}$$

[a] The electric potential at point $\mathbf{r}$ of point charge q at location $\mathbf{r}'$ is $kq/|\mathbf{r}-\mathbf{r}'|$

1.3. If a point charge Q is placed at rest at the origin, will it remain at rest? Justify using electric potential and symbolic computation.

The charge at the origin will stay at rest provided the gradient of the potential at the origin is zero because in that case, the electric field at the origin will be zero, and thus the force it feels at the origin will be zero. We need to take the gradient, plug in $(x, y) = (0, 0)$, and see if we get zero. Taking the gradient gives

$$\nabla V(x,y) = -kq\left[\frac{(x-d/2)\hat{\mathbf{x}} + (y-d\sqrt{3}/2)\hat{\mathbf{y}}}{[(x-d/2)^2 + (y-d\sqrt{3}/2)^2]^{3/2}} + \frac{(x-d/2)\hat{\mathbf{x}} + (y+d\sqrt{3}/2)\hat{\mathbf{y}}}{[(x-d/2)^2 + (y+d\sqrt{3}/2)^2]^{3/2}} + \frac{(x+d)\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{[(x+d)^2 + y^2]^{3/2}}\right] \tag{6}$$

So at the origin we get

$$\nabla V(0,0) = -kq\left[\frac{-d/2\hat{\mathbf{x}} - d\sqrt{3}/2\hat{\mathbf{y}}}{[(d/2)^2 + (d\sqrt{3}/2)^2]^{3/2}} + \frac{-d/2\hat{\mathbf{x}} + d\sqrt{3}/2\hat{\mathbf{y}}}{[(d/2)^2 + (d\sqrt{3}/2)^2]^{3/2}} + \frac{d\hat{\mathbf{x}}}{[d^2]^{3/2}}\right] \tag{7}$$

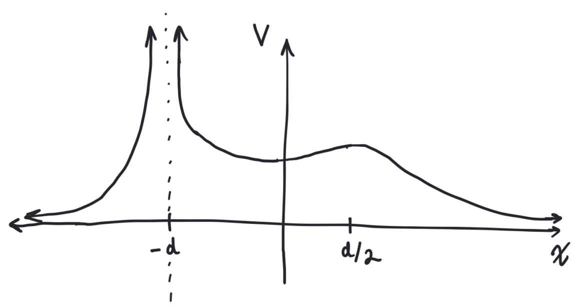
$$= -\frac{kq}{d^3}[(-d/2 - d/2 + d)\hat{\mathbf{x}} + (-d\sqrt{3}/2 + d\sqrt{3}/2)\hat{\mathbf{y}}] \tag{8}$$

$$= \mathbf{0} \tag{9}$$

So the charge will stay at the origin if placed there at rest.

1.4. Sketch the graph of $V(x,0)$ versus x.

Before we sketch the graph, let's make some physical observations that will help draw it. Since the charge distribution is bounded, we expect that if we go really far away, the potential will drop to zero. Since the charges are all of the same sign, the potential will be positive everywhere if $q > 0$ and negative everywhere if $q < 0$. Let's take $q > 0$ for concreteness, and note that for $q < 0$ the graph is just flipped. Since there is a point charge at $x = -d$, we expect there to be an asymptote there where the potential goes to infinity. The part that's harder to determine is what happens for positive values of x roughly between 0 and d. We know that close to the charge at $x = -d$, the potential is big, but then does it decreasing from there on out to infinity, or is there perhaps a bump somewhere when we get close to the other vertices. I'd expect that there will be a small bump, but the only way to be sure is to check the value of the derivative with respect to x in the vicinity of $x = d/2$ and see if there's a local maximum somewhere around there. This problem won't be graded in such detail that it's crucial to get this feature correct as long as the graph has a feature indicating that the other two point charges affect the field and make it drop off less slowly, but it turns out that there is in fact a bump in that vicinity:



1.5. Compute the Taylor expansion of $V(x,0)$ about $x = 0$ up to the term of order x^2.

To compute the Taylor series, we need the first and second derivative of $V(x,0)$ with respect to x evaluated at $x = 0$. The potential itself is

$$V(x,0) = kq\left[\frac{1}{\sqrt{(x-d/2)^2 + (d\sqrt{3}/2)^2}} + \frac{1}{\sqrt{(x-d/2)^2 + (d\sqrt{3}/2)^2}} + \frac{1}{\sqrt{(x+d)^2}}\right] \tag{10}$$

$$= kq\left[\frac{2}{\sqrt{x^2 - dx + d^2}} + \frac{1}{\sqrt{(x+d)^2}}\right] \tag{11}$$

The first derivative is

$$\frac{d}{dx}V(x,0) = -kq\left[\frac{2x - d}{[x^2 - dx + d^2]^{3/2}} + \frac{x+d}{((x+d)^2)^{3/2}}\right] \tag{12}$$

and the second derivative is

$$\frac{d^2}{dx^2}V(x,0) = -kq\left[\frac{2(x^2-dx+d^2)-(3/2)(2x-d)^2}{(x^2-dx+d^2)^{5/2}} - \frac{2(x+d)^2}{[(x+d)^2]^{5/2}}\right] \tag{13}$$

These expressions look complicated (and frankly they are a pain to compute by hand), but when you evaluate them at $x = 0$, a *lot* of stuff simplifies, and you get

$$V(0,0) = \frac{3kq}{d}, \qquad \frac{d}{dx}V(x,0)\bigg|_{x=0} = 0, \qquad \frac{d^2}{dx^2}V(x,0)\bigg|_{x=0} = \frac{3}{2}\frac{kq}{d^3} \tag{14}$$

So the desired Taylor expansion or order x^2 is

$$V(x,0) = \frac{3kq}{d} + \frac{3}{4}\frac{kq}{d^3}x^2 + \cdots \tag{15}$$

This shows that near the origin, where the higher-order terms will be negligible, the potential looks like a parabola, so there's a little "bowl" at the origin if $q > 0$ and a little hill at the origin if $q < 0$.

1.6. If a point charge Q is placed at the origin and then given a sufficiently small kick in the x-direction, will it remain in the vicinity fo the origin forever? Does it depend on the sign of Q? Does it matter if the kick is to the left or right? Justify all answers carefully.

In the vicinity of the origin, the potential is either concave up if $q > 0$ or concave down if $q < 0$ as we can see from its Taylor expansion. Therefore, if $q > 0$, a positive charge Q that's given a small kick will remain near the origin, oscillating around the local minimum of the potential, and if $q < 0$, a negative charge Q that's given a kick will remain near the origin, oscillating around the local maximum of the potential. It doesn't matter if the charge is kicked to the left or to the right – all that matters is that the charge will remain near the origin oscillating provided its charge Q has the same sign as q, and it's given a sufficiently small kick.

1.7. If there is a case where the charge Q will oscillates under a small push in the x-direction, determine the period of small oscillations if the charge in the center has mass m. If there is not such a case of oscillatory motion, explain how you know this.

Using the Taylor expansion, the force on the charge in the x-direction hear the origin for small x is

$$F_x(x,0) = QE_x(x,0) = -Q\frac{d}{dx}V(x,0) = -\frac{3}{2}\frac{kQq}{d^3}x \tag{16}$$

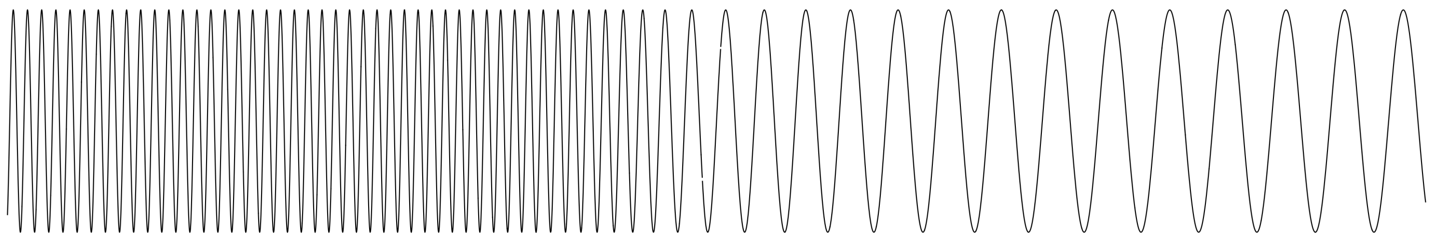
So by Newton's Second Law $F_x = m\ddot{x}$, we find that

$$\ddot{x} = -\frac{3}{2}\frac{kQq}{md^3}x \tag{17}$$

This is the harmonic oscillator equation $\ddot{x} = -\omega^2 x$ with $\omega = \sqrt{\frac{3}{2}\frac{kQq}{md^3}}$ which implies a period

$$T = 2\pi\sqrt{\frac{2}{3}\frac{md^3}{kQq}} \tag{18}$$

2. **Auditory Airplane Inference**. An airplane is flying past you some distance away at a constant speed in a straight line, and you use an app on your phone to record the sound it generates which has a constant emission frequency. The app outputs the following graph representing the pressure as a function of time in the air surrounding the phone. The graph displays 2 seconds of sensing data with the middle of the graph representing the moment of closest approach of the plane. Compute how fast the plane is going as accurately as you can, and clearly explain and show the logic and any algebra behind your computation.



We saw in the problem sets that even if the source of sound is not moving along the line joining the source to the observer, the asymptotic relationship between the observed and emitted frequencies is the same as it would be if the source *were* moving along that line. So when the plane is far from the observer, approaching, the observed and emitted frequencies are related by

$$f_{o,\text{toward}} \approx \frac{v}{v - v_s} f_s \tag{19}$$

while when the plane is far from the observer, moving away, the observed and emitted frequencies are related by

$$f_{o,\text{away}} \approx \frac{v}{v + v_s} f_s \tag{20}$$

This is a system of two equations in two unknowns, v_s, f_s, and we therefore can solve for both. Since we care about v_s, the speed of the source, we eliminate f_s by dividing the equations by each other and then solving for v_s. The result is

$$v_s = \frac{f_{o,\text{toward}} - f_{o,\text{away}}}{f_{o,\text{toward}} + f_{o,\text{away}}} v \tag{21}$$

If we now look at the waveform given in the problem statement, we can approximate $f_{o,\text{toward}}$ and $f_{o,\text{away}}$ by, for example, looking at roughly the first half-second in the diagram when the plane is a bit on the far side moving toward, and noting that there are about 22 cycles in that first half second which gives a frequency of 44 Hz. On the other hand, in the last half second there are about 6 full cycles giving a frequency of ~ 12 Hz. Also the speed of sound in air is 340 m/s, so we get the following approximation for the speed of the plane:

$$v_s \approx \frac{44\,\text{Hz} - 12\,\text{Hz}}{44\,\text{Hz} + 12\,\text{Hz}}(340\,\text{m/s}) \approx 194\,\text{m/s} \tag{22}$$

Note that this is an approximation, and one could get answers that are something like ±25% different than this one if one had counted the cycles slightly differently, which is fine. The cool thing is that this is even possible – simply by listening to something move by, you can estimate its speed using the doppler effect, even without knowing the frequency at which it's emitting sound!

3. True or False questions. Determine whether or not each of the following statements is true. If a statement is true, prove it. If the statement is false, provide a counterexample and explain how it constitutes a counterexample. Diagrams can be useful in explaining such things.

3.1. If the electric potential in a certain region of space is constant, then the charge enclosed by any closed surface completely contained within that region is zero.

True. If the electric potential is constant throughout a region, then since $\mathbf{E} - -\nabla V$, the electric field is zero throughout that region. It follows that the flux $\int \mathbf{E} \cdot \hat{\mathbf{n}} dA$ will be zero for any closed surface contained within the region, and by Gauss's Law, this means that the charge enclosed by the closed surface is zero.

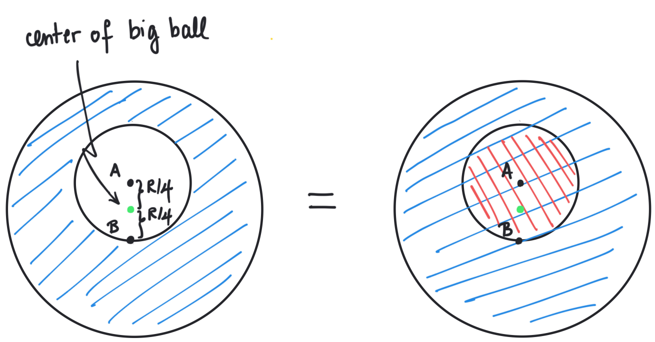
3.2. A sphere of radius R is centered at the origin. A total charge Q is uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the the origin is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.

True. By spherical symmetry, the electric field of the distribution must point either radially outward or radially inward everywhere. Now suppose we apply Gauss's Law to a Gaussian sphere of radius $r < R/2$ centered at the origin. The flux is, on one hand $E(r)(4\pi r^2$ where $E(r)\hat{\mathbf{n}}$ is the field at radius r, and on the other hand it's zero since no charge is enclosed by the surface. It follows that $E(r) = 0$, so the field is zero everywhere within the cavity. It follows that for any two points A and B in the cavity the difference in potential between them is zero;

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \tag{23}$$

Thus potential has the same value at every two such points.

3.3. A sphere of radius R is centered at the origin. A total charge Q uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the point $(x, y, z) = (0, 0, R/4)$ is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.



False. There is a trick to this problem. If you were to take the charge distribution consisting of the original ball without the cavity cut out, and then if you were to add to it a oppositely-charged ball having the same magnitude of its charge density, then this is equivalent as far as the potential is concerned to the ball with the cavity cut out because the opposite charges would cancel giving net zero charge in the region where they overlap. Now suppose that you consider two points A and B along the shared diameter line of the two balls, one in the center of the oppositely-charged (red) ball and one at its edge. Both of these points are $R/4$ from the center of the big ball, but they are at different points in the oppositely charged ball, so the potential due to the big ball will be the same at these points, but not due to the small ball, and therefore these are two points where the total potential due to the superposition of both balls will be different.