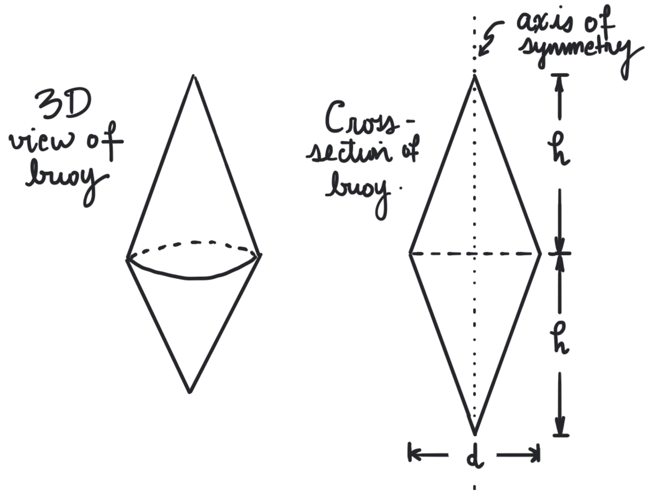
Instructions.

1. Complete this exam packet, and submit it to Gradescope before the end of the exam period.
2. Make sure all final work is in only the spaces provided, and make sure to upload a document of exactly the length of this packet.
3. If you are using a smartphone to generate a scan, please use a scanning app such as Adobe Scan to quickly generate an optimized PDF document.
4. There is extra space at the end of the packet in case the space below each problem isn't sufficient, but if you use that extra space, make sure to indicate that extra work is contained there when you show your work for a problem.
5. You are allowed to use the CCLE course page, the course textbook (OpenStax), and your notes, but you are not permitted to use any other internet resources.
6. A calculator (whatever type desired) is allowed.
7. You may not communicate about the contents of this exam with anyone during the exam period.
8. You may not logon to Campuswire during the exam period.
9. **Josh and the TAs will not be free to answer clarifying questions about the exam during the exam period**. If you believe there is an ambiguity, feel free to make a note of it to yourself and we will attempt to address it post-exam, but otherwise do your best to answer the questions based on what they say.
10. Violations of instructions pertaining to what you are not allowed to do during the exam will be considered cheating and will be reported to the Dean of Students.

1. Bobbing Buoy. A buoy is a floating device that can have many purposes, but often as a locator for ships. Collin constructs a hollow metal buoy by welding together two identical cones of height h and diameter d. The resulting double-cone buoy has average density equal to half the density of seawater meaning that its total mass divided by total volume is half the density of seawater. Throughout this problem, you can ignore the air above the sea's surface.

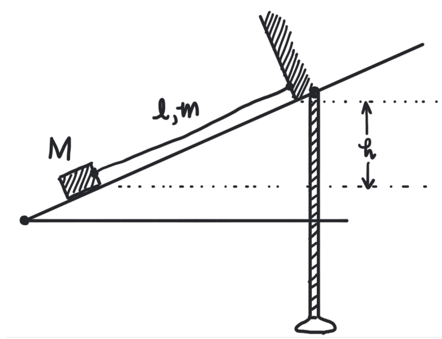


1.1. When the buoy is at rest in a calm ocean, with its axis of symmetry perpendicular to the water's surface, what fraction of it will be submerged?

1.2. If the buoy is at rest at time $t = 0$ and is then pushed down slightly into the water and let go, what will be its angular frequency ω of small oscillations in terms of the given variables? Note: the volume of a cone of height h and base area A is $hA/3$.

1.3. What should be the height h of each cone in the buoy so that the buoy will execute one oscillation period every second and therefore be usable as a clock with one-second accuracy?

2. Mills' Musical Machine. A Mills banjo is a musical instrument which relies on being placed near the surface of the Earth to operate. It consists of a block of mass M that is being prevented from sliding down a frictionless incline by a thin string of length ℓ and mass m attached to a stationary wall. The steepness of the incline can be adjusted by adjusting the height h with a screw as indicated on the diagram. The string can be plucked like a guitar string. Assume that the block is sufficiently massive that the string can be treated as though it's fixed at both endpoints.

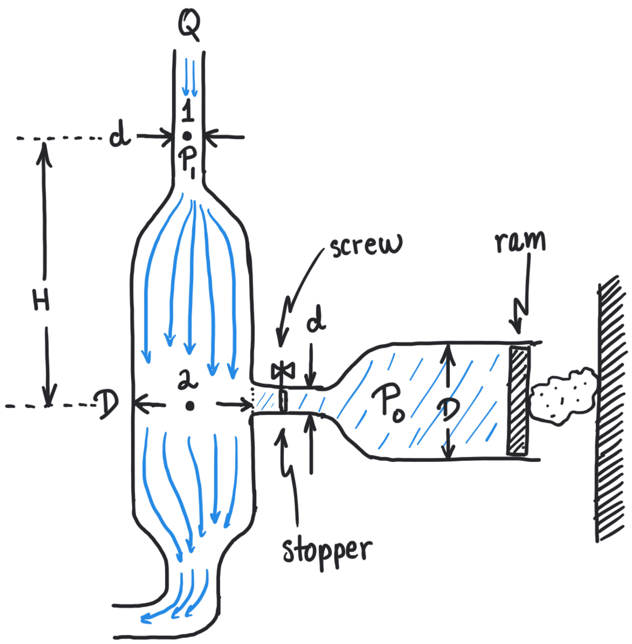


2.1. If all other variables besides h are held fixed, do you expect the frequencies of the harmonics on the string to increase, stay the same, or decrease if h is increased? Make as compelling a physical argument as you can without writing down any equations.

2.2. What is the frequency of the n^{th} harmonic of the string in terms of the variables given in the problem statement?

2.3. If the block has a mass of $20\,\text{kg}$, the height h is $0.5\,\text{m}$, the string has a length of $0.77\,\text{m}$, and the 5th harmonic has frequency $445\,\text{Hz}$, what is the linear mass density μ of the string?

3. In the apparatus below, the Dynamic Aqua Crusher, water is injected into a vertical channel at a volume flow rate Q. The diameter of the channel at point 1 is d, and its diameter at point 2 is D. To the right of point 2 is a horizontal channel blocked by a stopper. The small diameter of this horizontal channel is d, and the large diameter is D. To the right of the stopper is a chamber with static water. When the stopper is clamped in place by a screw, the pressure in the chamber is p_0. When the clamp is unscrewed, the stopper is free to move. At the right-hand end of the chamber is a ram that can be used to crush things against a wall. Let ρ be the density of water, p_1 be the pressure at point 1 in the vertical channel, and H be the vertical distance between points 1 and 2. You may assume that $p_1 > p_0$.



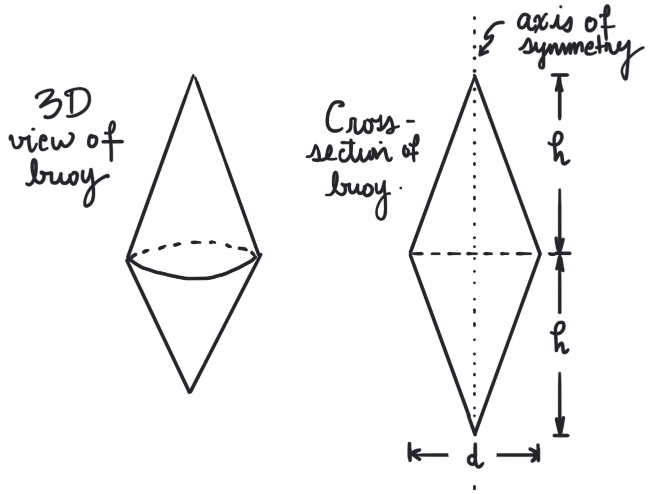
3.1. Suppose that the clamp is unscrewed and the stopper is free to move. Find an expression in terms of the given variables for the force exerted by the ram on whatever it's crushing in this circumstance.

3.2. If $d = 1\,\text{cm}$, $D = 5\,\text{cm}$, $H = 1\,\text{m}$, $p_1 = 2 \times 10^5\,\text{Pa}$, $p_0 = 10^5\,\text{Pa}$, and $Q = 100\,\text{cm}^3/\text{s}$, what is the magnitude of the force that the ram exerts on the object it's crushing?

Instructions.

1. Complete this exam packet, and submit it to Gradescope before the end of the exam period.
2. Make sure all final work is in only the spaces provided, and make sure to upload a document of exactly the length of this packet.
3. If you are using a smartphone to generate a scan, please use a scanning app such as Adobe Scan to quickly generate an optimized PDF document.
4. There is extra space at the end of the packet in case the space below each problem isn't sufficient, but if you use that extra space, make sure to indicate that extra work is contained there when you show your work for a problem.
5. You are allowed to use the CCLE course page, the course textbook (OpenStax), and your notes, but you are not permitted to use any other internet resources.
6. A calculator (whatever type desired) is allowed.
7. You may not communicate about the contents of this exam with anyone during the exam period.
8. You may not logon to Campuswire during the exam period.
9. **Josh and the TAs will not be free to answer clarifying questions about the exam during the exam period**. If you believe there is an ambiguity, feel free to make a note of it to yourself and we will attempt to address it post-exam, but otherwise do your best to answer the questions based on what they say.
10. Violations of instructions pertaining to what you are not allowed to do during the exam will be considered cheating and will be reported to the Dean of Students.

1. **Bobbing Buoy**. A buoy is a floating device that can have many purposes, but often as a locator for ships. Collin constructs a hollow metal buoy by welding together two identical cones of height h and diameter d. The resulting double-cone buoy has average density equal to half the density of seawater meaning that its total mass divided by total volume is half the density of seawater. Throughout this problem, you can ignore the air above the sea's surface.



1.1. When the buoy is at rest in a calm ocean, with its axis of symmetry perpendicular to the water's surface, what fraction of it will be submerged?

Let M be the mass of the buoy. It is specified that the average density of the buoy equals half that of seawater. Letting the density of seawater be ρ, this implies $M/V = \rho/2$ and therefore $M = \rho V/2$ so that its weight is $\rho V g/2$. On the other hand, if $V_{\text{submerged}}$ is the submerged volume, then the magnitude ofhte buoyant force on the buoy is $\rho V_{\text{submerged}} g$. Combining Archimedes' principle and Newton's Second Law in the vertical direction to when the buoy is in static equilibrium, we therefore obtain

$$\rho V_{\text{submerged}} g - \rho V g/2 = 0 \tag{1}$$

Therefore

$$V_{\text{submerged}} = \frac{V}{2}. \tag{2}$$

Half of the buoy's volume will be submerged below the water.

1.2. If the buoy is at rest at time $t = 0$ and is then pushed down slightly into the water and let go, what will be its angular frequency ω of small oscillations in terms of the given variables? Note: the volume of a cone of height h and base area A is $hA/3$.

Let the positive z-direction point vertically upward perpendicular to the surface of the water. Let $z = 0$ be the position of a point in the middle of the buoy. The volume of water displaced by the buoy depends on z, and we call it $V(z)$. This volume is given by the $V/2$ as our previous static equilibrium analysis revealed, plus the volume of a sliver of one of the cones starting at its base and having a height z. Let's call the (signed – can be positive or negative) volume of this sliver $\Delta V(z)$, so we have $V(z) = V/2 + \Delta V(z)$. Applying's Newton's Second Law and Archimedes' principle in the z-direction gives

$$M\ddot{z} = \rho(V/2 + \Delta V(z))g - Mg \tag{3}$$

Recall that $M = \rho V/2$ based on the problem statement. Plugging this observation into the Newton's Second Law equation above, and doing a little simplifying yields

$$(V/2)\ddot{z} = \Delta V(z)g \tag{4}$$

It remains to find an expression for $\Delta V(z)$. This is given by -1 times the difference between the volume of one of the cones in the buoy minus the volume of the sub-cone of height $h - z$. The -1 factor takes into consideration the fact that when z is positive $\Delta V(z)$ will be negative since less volume will be submerged, and vice versa. Using similar triangles, we find that the sub-cone with height h has a base diameter $d(z)$ given by

$$\frac{d(z)}{d} = \frac{h - z}{h} \tag{5}$$

Putting all of these observations together gives

$$\Delta V(z) = -\left[\frac{1}{3}\pi(d/2)^2 h - \frac{1}{3}\pi(d(z)/2)^2(h - z)\right] \tag{6}$$

$$= -\left[\frac{1}{3}\pi\left(\frac{d}{2}\right)^2 h - \frac{1}{3}\pi\left(\frac{h - z}{h}\frac{d}{2}\right)^2 (h - z)\right] \tag{7}$$

$$= -\frac{1}{3}\pi\left(\frac{d}{2}\right)^2 h\left[1 - \left(\frac{h - z}{h}\right)^3\right] \tag{8}$$

$$= -(V/2)\left[1 - \left(1 - \frac{z}{h}\right)^3\right] \tag{9}$$

Plugging this back into the Newton's Second Law expression (4) gives

$$\ddot{z} = -\left[1 - \left(1 - \frac{z}{h}\right)^3\right] g \tag{10}$$

This isn't the harmonic oscillator equation, but if we make a small-z approximation, we find that $(1 - z/h)^3 = 1 - 3(z/h) + \cdots$. Plugging this into the equation of motion and only retaining the lowest non-vanishing term gives

$$\ddot{z} \approx -(3g/h)z. \tag{11}$$

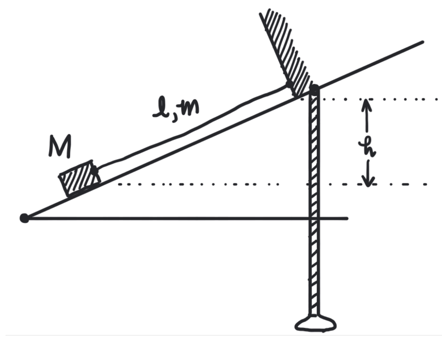
This is now the form of the harmonic oscillator equation, and we can immediately identify the angular frequency of small oscillations as $\omega = \sqrt{3g/h}$.

1.3. What should be the height h of each cone in the buoy so that the buoy will execute one oscillation period every second and therefore be usable as a clock with one-second accuracy?

Recall that oscillation period is related to frequency as $T = 1/f$, and angular frequency is related to frequency as $\omega = 2\pi f$, so period is related to angular frequency as $\omega = 2\pi/T$. Plugging in our expression for ω in terms of the buoy's cone height, solving for h, and inputting $1\,\mathrm{s}$ for T gives

$$h = 3g\left(\frac{T}{2\pi}\right)^2 \approx 3(9.8\,\mathrm{m/s^2})\left(\frac{1\,\mathrm{s}}{2\pi}\right)^2 \approx 0.74\,\mathrm{m} \tag{12}$$

2. Mills' Musical Machine. A Mills banjo is a musical instrument which relies on being placed near the surface of the Earth to operate. It consists of a block of mass M that is being prevented from sliding down a frictionless incline by a thin string of length ℓ and mass m attached to a stationary wall. The steepness of the incline can be adjusted by adjusting the height h with a screw as indicated on the diagram. The string can be plucked like a guitar string. Assume that the block is sufficiently massive that the string can be treated as though it's fixed at both endpoints.



2.1. If all other variables besides h are held fixed, do you expect the frequencies of the harmonics on the string to increase, stay the same, or decrease if h is increased? Make as compelling a physical argument as you can without writing down any equations.

If h is increased, then the slope on which the block sits is increased in steepness, and this will increase the tension in the string. Given that the linear mass density of the string doesn't change, this in turn implies that the wave speed along the string increases. Now recall that the wavelengths of the harmonics are fixed by the length of the string itself as a consequence of the fixed endpoint condition, so increasing h does not change the wavelengths of the harmonics. How can it be that the wavelengths of the harmonics stay the same but the wave speed along the string has increased? This is only possible of the frequencies of the harmonics increase. Why is this so? Well harmonics can be thought of as summing traveling waves moving in opposite directions and the wavelength tells you how far they travel each period of oscillation. To increase in speed with a fixed wavelength, the period needs to therefore decrease, and this corresponds to an increase in frequency.

2.2. What is the frequency of the n^{th} harmonic of the string in terms of the variables given in the problem statement?

Newton's Second Law applied to static equilibrium of the block and in the direction of the slope reveals that $F_T - Mg\sin\theta = 0$, where θ is the angle between the slope and the horizontal direction. On the other hand, trigonometry reveals that $\sin\theta = h/\ell$. Combining these facts with the fact that the wave speed along the string is $v = \sqrt{F_T/\mu}$ and the frequency of the n^{th} harmonic is $f_n = nv/2\ell$ gives

$$f_n = \frac{n}{2\ell}\sqrt{\frac{Mgh/\ell}{m/\ell}} = \frac{n}{2\ell}\sqrt{\frac{M}{m}gh} \tag{13}$$

2.3. If the block has a mass of 20 kg, the height h is 0.5 m, the string has a length of 0.77 m, and the 5th harmonic has frequency 445 Hz, what is the linear mass density μ of the string?

The calculations in the last part of the problem show that the linear mass density $\mu = m/\ell$ satisfies

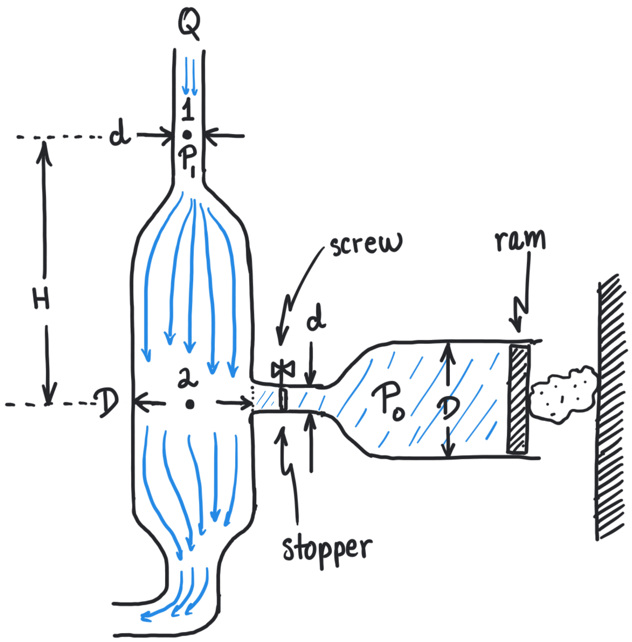
$$\mu = \frac{Mgh}{\ell}\left(\frac{n}{2\ell f_n}\right)^2 = \frac{Mghn^2}{4\ell^3 f_n^2} \tag{14}$$

Plugging in the particular values specified gives

$$\mu \approx \frac{(20\,\text{kg})(9.8\,\text{m/s}^2)(0.5\,\text{m})(5^2)}{4(0.77\,\text{m})^3(445\,\text{Hz})^2} \approx 0.007\,\text{kg/m} = 7\,\text{g/m} \tag{15}$$

This is roughly the linear mass density of a guitar string.

3. In the apparatus below, the Dynamic Aqua Crusher, water is injected into a vertical channel at a volume flow rate Q. The diameter of the channel at point 1 is d, and its diameter at point 2 is D. To the right of point 2 is a horizontal channel blocked by a stopper. The small diameter of this horizontal channel is d, and the large diameter is D. To the right of the stopper is a chamber with static water. When the stopper is clamped in place by a screw, the pressure in the chamber is p_0. When the clamp is unscrewed, the stopper is free to move. At the right-hand end of the chamber is a ram that can be used to crush things against a wall. Let ρ be the density of water, p_1 be the pressure at point 1 in the vertical channel, and H be the vertical distance between points 1 and 2. You may assume that $p_1 > p_0$.



3.1. Suppose that the clamp is unscrewed and the stopper is free to move. Find an expression in terms of the given variables for the force exerted by the ram on whatever it's crushing in this circumstance.

Let $+z$ point upward from point 2 to point 1 in the diagram with $z = 0$ at point 2. Consider the period of time before the stopper is unscrewed. Conservation of volume flow for an incompressible fluid gives $Q = v_1\pi(d/2)^2 = v_2\pi(D/2)^2$. Combining this with Bernoulli's Principle applied to a streamline from point 1 to point 2 yields

$$p_1 + \rho g H + \frac{1}{2}\rho\left(\frac{Q}{\pi(d/2)^2}\right)^2 = p_2 + \frac{1}{2}\rho\left(\frac{Q}{\pi(D/2)^2}\right)^2 \tag{16}$$

Solving for p_2 gives

$$p_2 = p_1 + \rho g H + \frac{8\rho Q^2}{\pi^2 d^4}\left(1 - (d/D)^4\right) \tag{17}$$

Notice that since $d < D$, this result implies that $p_2 > p_1$. If we combine this inequality with $p_1 > p_0$ given in the problem specification, we find that $p_2 > p_0$. In other words, before the stopper is unscrewed, the water to the left of the stopper exerts more pressure on the stopper than the water to its right. This means, by Newton's Second Law, that the screw must be applying an additional force to the left that holds it in place. Then the screw is removed, the stopper must still remain stationary despite the initial differential in pressures on its two sides because there is an incompressible fluid (water) in the chamber to its right, but this means that the water to its right must apply an additional pressure $p_2 - p_0$ on the stopper to maintain static equilibrium. It follows that the stopper exerts an additional pressure $p_2 - p_0$ on the water in the chamber that is in contact with the stopper, but Pascal's Principle then implies that this additional pressure is added uniformly to *all* points in the chamber. As a result, the final pressure everywhere in the chamber is $p_0 + (p_2 - p_0) = p_2$. This means that with the stopper unscrewed, the force of the water on the ram, and hence the magnitude of the force it exerts on whatever it's crushing, will be $p_2\pi(D/2)^2$. Plugging in our expression for p_2 from above gives

$$F_{\text{ram}} = \pi(D/2)^2\left[p_1 + \rho g H + \frac{8\rho Q^2}{\pi^2 d^4}\left(1 - (d/D)^4\right)\right] \tag{18}$$

Notice, in particular, that p_0 does not appear anywhere in this expression.

3.2. If $d = 1\,\text{cm}$, $D = 5\,\text{cm}$, $H = 1\,\text{m}$, $p_1 = 2\times10^5\,\text{Pa}$, $p_0 = 10^5\,\text{Pa}$, and $Q = 100\,\text{cm}^3/\text{s}$, what is the magnitude of the force that the ram exerts on the object it's crushing?

Plugging in the given values gives

$$F_{\text{ram}} = \pi(2.5\times10^{-2}\,\text{m})^2\left[2\times10^5\,\text{Pa} + (1000\,\text{kg/m}^3)(9.8\,\text{m/s}^2)(1\,\text{m}) + \frac{8(1\,\text{kg/m}^2)(10^{-4}\,\text{m}^3/\text{s})^2}{\pi^2(10^{-2}\,\text{m})^4}(1-(1/5)^4)\right] \tag{19}$$

$$\approx \underbrace{392.70\,\text{N}}_{\text{force due to } p_1} + \underbrace{19.24\,\text{N}}_{\text{force due to change in height}} + \underbrace{0.0016\,\text{N}}_{\text{force due to change in speed}} \tag{20}$$

$$\approx 412\,\text{N}. \tag{21}$$

It's interesting to note that with these values, the extra force contributed by the change in height and change in speed of the fluid flow are basically negligible compared to the force due to the pressure p_1 applied to the flow in the first place. However, one can cause the additional force due to the change in flow speed to dominate the other terms by making d much smaller.

Space for extra work.

Space for extra work.