Midterm 2

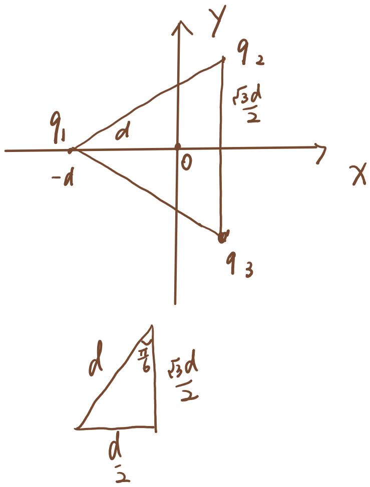
Tuesday, May 19, 2020 12:03 AM

nstructions.

1. Complete this exam packet, and submit it to Gradescope before the end of the exam period.
2. Make sure all final work is in only the spaces provided, and make sure to upload a document of exactly the length of this packet.
3. If you are using a smartphone to generate a scan, please use a scanning app such as Adobe Scan to quickly generate an optimized PDF document.
4. There is extra space at the end of the packet in case the space below each problem isn't sufficient, but if you use that extra space, make sure to indicate that extra work is contained there when you show your work for a problem.
5. You are allowed to use the CCLE course page, the course textbook (OpenStax), and your notes, but you are not permitted to use any other internet resources.
6. A calculator (whatever type desired) is allowed.
7. You may not communicate about the contents of this exam with anyone during the exam period.
8. You may not logon to Campuswire during the exam period.
9. **Josh and the TAs will not be free to answer clarifying questions about the exam during the exam period**. If you believe there is an ambiguity, feel free to make a note of it to yourself and we will attempt to address it post-exam, but otherwise do your best to answer the questions based on what they say.
10. Violations of instructions pertaining to what you are not allowed to do during the exam will be considered cheating and will be reported to the Dean of Students.

. **Treacherous Triangle Trickery**. Consider a charge distribution consisting of an equilateral triangle with a oint charge q fixed at each of its vertices. Let d be the distance between the center of the triangle and each ertex, let the triangle's center be at the origin, and let one of its vertices lie on the x-axis at the point $x = -d$.

.1. Compute the electric field at the center of the triangle by explicitly computing the sum of the electric fields ue the charges at each vertex.



$$\vec{E} = \vec{E}(\vec{r}_1) + \vec{E}(\vec{r}_2) + \vec{E}(\vec{r}_3)$$

$\vec{r} = (0,0)$ $\quad$ $\vec{r}_2 = (\frac{d}{2}, \frac{\sqrt{3}d}{2})$

$\vec{r}_1 = (-d,0)$ $\quad$ $\vec{r}_3 = (\frac{d}{2}, -\frac{\sqrt{3}d}{2})$

$$\vec{E}(\vec{r}_1) = \frac{kq}{|\vec{r}-\vec{r}_1|^2}\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|} = \frac{kq}{d^2}\frac{(d,0)}{d} = \frac{kq}{d^2}(1,0)$$

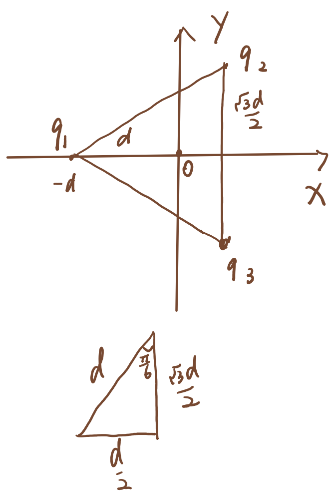
$$\vec{E}(\vec{r}_2) = \frac{kq}{|\vec{r}-\vec{r}_2|^2}\frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|} = \frac{kq}{d^2}\frac{(-\frac{d}{2}, -\frac{\sqrt{3}d}{2})}{d} = \frac{kq}{d^2}(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

$$\vec{E}(\vec{r}_3) = \frac{kq}{|\vec{r}-\vec{r}_3|^2}\frac{\vec{r}-\vec{r}_3}{|\vec{r}-\vec{r}_3|} = \frac{kq}{d^2}\frac{(-\frac{d}{2}, \frac{\sqrt{3}d}{2})}{d} = \frac{kq}{d^2}(-\frac{1}{2}, \frac{\sqrt{3}}{2})$$

$$\therefore \vec{E}(\vec{r}) = \vec{E}(\vec{r}_1) + \vec{E}(\vec{r}_2) + \vec{E}(\vec{r}_3)$$

$$= \frac{kq}{d^2}\left((1,0) + (-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}) + (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\right) = \boxed{0}$$

.2. Let $V(x, y)$ be the electric potential as a function of position. Compute an expression for $V(x, y)$, and try to implify it if possible.



let $\vec{r} = (x,y)$

$\vec{r}_1 = (-d,0)$, $\vec{r}_2 = (\frac{d}{2}, \frac{\sqrt{3}d}{2})$, $\vec{r}_3 = (\frac{d}{2}, -\frac{\sqrt{3}d}{2})$

$$V(x,y) = \sum_{i=1}^{3} \frac{kq}{|\vec{r}-\vec{r}_i|}$$

$$= \frac{kq}{\sqrt{(x+d)^2+y^2}} + \frac{kq}{\sqrt{(x-\frac{d}{2})^2+(y-\frac{\sqrt{3}d}{2})^2}} + \frac{kq}{\sqrt{(x-\frac{d}{2})^2+(y+\frac{\sqrt{3}d}{2})^2}}$$

$$\boxed{= kq\left(\frac{1}{\sqrt{(x+d)^2+y^2}} + \frac{1}{\sqrt{(x-\frac{d}{2})^2+(y-\frac{\sqrt{3}}{2}d)^2}} + \frac{1}{\sqrt{(x-\frac{d}{2})^2+(y+\frac{\sqrt{3}d}{2})^2}}\right)}$$

.3. If a point charge Q is placed at rest at the origin, will it remain at rest? Justify using electric potential and ymbolic computation.

According to 2, $V(x,y) = kq\left(\frac{1}{\sqrt{(x+d)^2+y^2}} + \frac{1}{\sqrt{(x-\frac{d}{2})^2+(y-\frac{\sqrt{3}d}{2})^2}} + \frac{1}{\sqrt{(x-\frac{d}{2})^2+(y+\frac{\sqrt{3}}{2}d)^2}} \right)$

$$= kq\left(\left((x+d)^2+y^2\right)^{-\frac{1}{2}} + \left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{1}{2}} + \left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{1}{2}} \right)$$

$$\therefore \frac{dV(x,y)}{dx} = kq\left(-\tfrac{1}{2}\left((x+d)^2+y^2\right)^{-\frac{3}{2}} \cdot 2(x+d) + (-\tfrac{1}{2})\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}} \cdot 2(x-\tfrac{d}{2}) + (-\tfrac{1}{2})\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}} \cdot 2(x-\tfrac{d}{2}) \right)$$

$$\therefore \frac{dV(x,y)}{dy} = kq\left(-\tfrac{1}{2}\left((x+d)^2+y^2\right)^{-\frac{3}{2}} \cdot 2y + (-\tfrac{1}{2})\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}} \cdot 2(y-\tfrac{\sqrt{3}d}{2}) + (-\tfrac{1}{2})\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}} \cdot 2(y+\tfrac{\sqrt{3}d}{2}) \right)$$

$$\therefore \frac{dV(0,0)}{dx} = kq\left(-\tfrac{1}{2}d^{-3} \cdot 2d + (-\tfrac{1}{2})\left(\tfrac{d^2}{4}+\tfrac{3d^2}{4}\right)^{-\frac{3}{2}}(-d) + (-\tfrac{1}{2})\left(\tfrac{d^2}{4}+\tfrac{3d^2}{4}\right)^{-\frac{3}{2}}(-d) \right) = kq\left(-\tfrac{1}{2}d^{-3}(2d-d-d)\right) = 0$$

$$\frac{dV(0,0)}{dy} = kq\left(0 + (-\tfrac{1}{2})\left(\tfrac{d^2}{4}+\tfrac{3d^2}{4}\right)^{-\frac{3}{2}} \cdot (-\sqrt{3}d) + (-\tfrac{1}{2})\left(\tfrac{d^2}{4}+\tfrac{3d^2}{4}\right)^{-\frac{3}{2}}(\sqrt{3}d) \right) = kq\left(-\tfrac{1}{2}d^{-3}(-\sqrt{3}d+\sqrt{3}d)\right) = 0$$

$\therefore \vec{\nabla} V(0,0) = 0$

$\therefore \vec{F} = -\vec{\nabla} V(0,0) = 0$

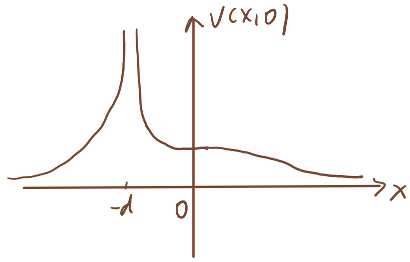
Q is placed at rest at origin, so it will stay at rest unless acted upon by an unbalanced forced. Since $\vec{F}(0,0)=0$, so no force will be acted on Q

$\therefore Q$ will remain $\boxed{\text{at rest}}$

.4. Sketch the graph of $V(x,0)$ versus x.

$$V(x,0) = kq\left(\frac{1}{|x+d|} + \frac{1}{\sqrt{(x-\frac{d}{2})^2 + \frac{3}{4}d^2}} + \frac{1}{\sqrt{(x-\frac{d}{2})^2+\frac{3}{4}d^2}} \right)$$

$$= kq\left(\frac{1}{|x+d|} + \frac{2}{\sqrt{(x-\frac{d}{2})^2+\frac{3}{4}d^2}} \right)$$



.5. Compute the Taylor expansion of $V(x,0)$ about $x = 0$ up to the term of order x^2.

$$V(x,0) = kq\left(\frac{1}{x+d} + \frac{2}{\sqrt{(x-\frac{d}{2})^2+\frac{3}{4}d^2}} \right) = kq\left((x+d)^{-1} + 2(x^2+d^2-xd)^{-\frac{1}{2}} \right), \quad V(0,0) = kq\cdot 3d^{-1}$$

$$V'(x,0) = kq\left(-(x+d)^{-2} + (-1)(x^2+d^2-xd)^{-\frac{3}{2}}(2x-d) \right) \quad V'(0,0) = kq\left(-d^{-2} + (-1)d^{-3}(-d)\right) = 0$$

$$V''(x,0) = kq\left(2(x+d)^{-3} + \frac{3}{2}(x^2+d^2-xd)^{-\frac{5}{2}}(2x-d)^2 \right), \quad V''(0,0) = kq\left(2d^{-3} + \left(\frac{3}{2}\right)\cdot d^{-5}\cdot d^2\right) = kq\left(\frac{3}{2}d^{-3}\right)$$

$$V(x,0) = \frac{V(0,0)}{0!} + \frac{V'(0,0)x}{1!} + \frac{V''(0,0)x^2}{2!}$$

$$\boxed{= \frac{3kq}{d} + \frac{3kq\,x^2}{4d^3}}$$

.6. If a point charge Q is placed at the origin and then given a sufficiently small kick in the x-direction, will it emain in the vicinity fo the origin forever? Does it depend on the sign of Q? Does it matter if the kick is to the eft or right? Justify all answers carefully.

Yes No

$\frac{dV(0,0)}{dx} = 0, \ \frac{dV(0,0)}{dy} = 0 \quad \therefore (0,0)$ is a critical point

$$\therefore \frac{dV(x,y)}{dx} = kq\left(-\tfrac{1}{2}\left((x+d)^2+y^2\right)^{-\frac{3}{2}}\cdot 2(x+d) + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\left(x-\tfrac{d}{2}\right) + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\left(x-\tfrac{d}{2}\right)\right)$$

$$\therefore \frac{dV(x,y)}{dy} = kq\left(-\tfrac{1}{2}\left((x+d)^2+y^2\right)^{-\frac{3}{2}}\cdot 2y + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\left(y-\tfrac{\sqrt{3}d}{2}\right) + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\left(y+\tfrac{\sqrt{3}d}{2}\right)\right)$$

$$\therefore \frac{dV^2(x,y)}{dx^2} = kq\left(\tfrac{3}{4}\left((x+d)^2+y^2\right)^{-\frac{5}{2}}\cdot 4(x+d)^2 + \left(-\tfrac{1}{2}\right)\left((x+d)^2+y^2\right)^{-\frac{3}{2}}\cdot 2 + \tfrac{3}{4}\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{5}{2}}\cdot 4\left(x-\tfrac{d}{2}\right)^2 + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\right.$$
$$\left. + \tfrac{3}{4}\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{5}{2}} 4\left(x-\tfrac{d}{2}\right)^2 + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\right)$$

$$\therefore \frac{dV^2(x,y)}{dy^2} = kq\left(\tfrac{3}{4}\left((x+d)^2+y^2\right)^{-\frac{5}{2}}\cdot 4y^2 + \left(-\tfrac{1}{2}\right)\left((x+d)^2+y^2\right)^{-\frac{3}{2}}\cdot 2 + \tfrac{3}{4}\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{5}{2}}\cdot 4\left(y-\tfrac{\sqrt{3}d}{2}\right)^2 + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y-\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\right.$$
$$\left. + \tfrac{3}{4}\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{5}{2}} 4\left(y+\tfrac{\sqrt{3}d}{2}\right)^2 + \left(-\tfrac{1}{2}\right)\left((x-\tfrac{d}{2})^2+(y+\tfrac{\sqrt{3}d}{2})^2\right)^{-\frac{3}{2}}\cdot 2\right)$$

$$\frac{dV^2(0,0)}{dx^2} = 3kq\left(\tfrac{3}{4}d^{-5}\cdot 4d^2 + (-1)d^{-3}\right) = 3kq\left(3d^{-3}-d^{-3}\right) = 3kq\left(2d^{-3}\right) > 0, \quad \frac{dV^2(0,0)}{dy} = 3kq\left(2d^{-3}\right) > 0$$

$\therefore V(0,0)$ is the local minimum.

① if $Q>0$, then Q tend to go down the electric potential, then Q will osicillate around origin since Q will always return to local minimum after kick. Q will remain in vicinity of $(0,0)$

② if $Q<0$, then Q tend to go up the electric potential, then Q will go up along the electric potential after being kicked, and will not remain in vicinity of origin

It does not depend on kick left or right because $V(0,0)$ is a critical point and is the local minimum in every direction

.7. If there is a case where the charge Q will oscillates under a small push in the x-direction, determine the period f small oscillations if the charge in the center has mass m. If there is not such a case of oscillatory motion, explain ow you know this.

Based on 1.5 Taylor expansion,

$V(x,0) = \frac{3kq}{d} + \frac{3kq}{2d^3}$

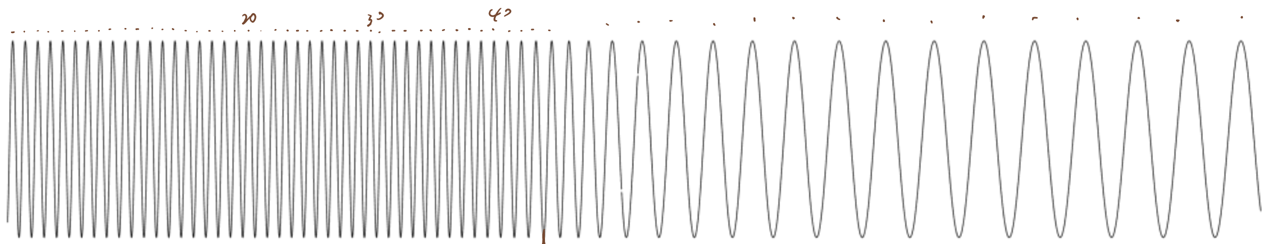
$\vec{E} = -\vec{\nabla} V(x,0) = \frac{3kq}{2d^3} x$

$\vec{F} = \vec{E} Q = \frac{3kqQ}{2d^3} x = m\ddot{x}$

$\therefore \ddot{x} = \frac{3kqQ}{2d^3 m} x \quad \therefore w = \sqrt{\frac{3kqQ}{2d^3 m}}$

$\boxed{T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{2md^3}{3kQq}}}$

. **Auditory Airplane Inference**. An airplane is flying past you some distance away at a constant speed in a traight line, and you use an app on your phone to record the sound it generates which has a constant emission requency. The app outputs the following graph representing the pressure as a function of time in the air surounding the phone. The graph displays 2 seconds of sensing data with the middle of the graph representing the noment of closest approach of the plane. Compute how fast the plane is going as accurately as you can, and learly explain and show the logic and any algebra behind your computation.



Sol= By dopplereffect, $f = \frac{v}{v \pm v_s} f_p$, let $v = 340\,m/s$,

① when plane approaching the person. ② when the plane fly away from the person

$f_1 = \frac{1}{T} = \frac{1}{(1/45)s} = 45\,Hz \qquad f_2 = \frac{1}{T} = \frac{1}{(1/15)s} = 15\,Hz.$

$f_1 = \left(\frac{v}{v - v_s}\right) f_p = 45\,Hz \qquad f_2 = \left(\frac{v}{v + v_s}\right) f_p = 15\,Hz.$

$\therefore \frac{f_1}{f_2} = \frac{\frac{v}{v - v_s}}{\frac{v}{v + v_s}} = \frac{v + v_s}{v - v_s} = \frac{45\,Hz}{15\,Hz} = 3$

$\therefore \quad V+V_S = 3V - 3V_S$

$4V_S = 2V$

$\therefore V = 2V_S$

let $V = 340 \, m/s$

$V_S \approx 170 \, m/s$

$\therefore$ speed of plane is about $\boxed{170 \, m/s}$

(Because the plane is very far away, we can roughly seen it as doppler effect)

. **True or False questions.** Determine whether or not each of the following statements is true. If a statement is rue, prove it. If the statement is false, provide a counterexample and explain how it constitutes a counterexample. Diagrams can be useful in explaining such things.

.1. If the electric potential in a certain region of space is constant, then the charge enclosed by any closed surface ompletely contained within that region is zero.

$\boxed{\text{True}}$

$\because$ V is constant

$\vec{E} = \vec{\nabla} V = \frac{dV}{d\vec{r}}$

$\because$ By Gauss law

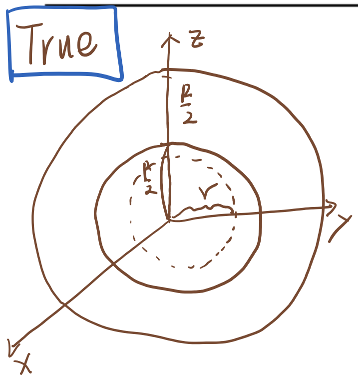
$\vec{E} = \frac{Q_{inside}}{\varepsilon_0} = \vec{\nabla} V = \frac{dV}{d\vec{r}} = 0$

$\because$ $\varepsilon_0 \neq 0$

$\therefore$ Q inside $= 0$

.2. A sphere of radius R is centered at the origin. A total charge Q is uniformly distributed throughout its nterior. A ball of radius $R/2$ centered at the the origin is carved out and discarded, leaving behind an empty avity. The electric potential has the same value at every point inside the cavity.

$\boxed{\text{True}}$



If we take Gauss surface of a sphere inside cavity, centered at origin, radius be r

By Gauss's law, $\vec{E}_{inside} = \frac{Q_{inside}}{\varepsilon_0} = \frac{0}{\varepsilon_0} = 0$

$\because$ $\vec{E} = \vec{\nabla}(V) = \frac{dV}{d\vec{r}}$

$\therefore$ V must be constant so that its deviative can be 0

$\therefore$ the electric potential has the same value at every point inside the cavity

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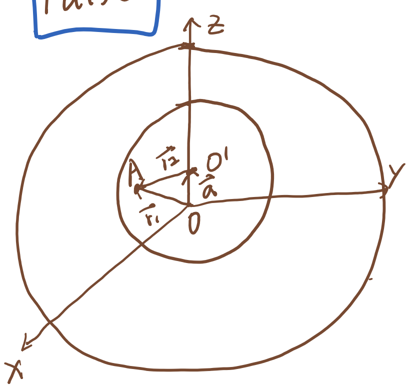
.3. A sphere of radius R is centered at the origin. A total charge Q uniformly distributed throughout its interior. A ball of radius $R/2$ centered at the point $(x, y, z) = (0, 0, R/4)$ is carved out and discarded, leaving behind an empty cavity. The electric potential has the same value at every point inside the cavity.

$\boxed{\text{False}}$



Consider a random point A inside the cavity.

let $\vec{OA} = \vec{r}_1$, $\vec{O'A} = \vec{r}_2$, $\vec{OO'} = \vec{a}$

Then $E_{big} = \frac{Q}{\varepsilon_0}$,

$\therefore \oint E_{big}\, dA = \frac{Q}{\varepsilon_0} = E_{big} 4\pi R^2 = \frac{Q}{\varepsilon_0}$

$\therefore E_{big} = \frac{Q}{4\pi R^2 \varepsilon_0}$, $\vec{E}_{big} = \frac{Q\vec{r}_1}{4\pi R^3 \varepsilon_0}$

Similarly, $\oint E_{small} = \frac{-Q}{\varepsilon_0} = E_{small} 4\pi \left(\frac{R}{2}\right)^2$

$\vec{E}_{small} = \frac{\rho \vec{r}_2}{3\varepsilon_0} = \frac{Q\vec{r}_2}{4\pi \varepsilon_0 R^3}$

$\therefore \vec{E}_{inside} = \vec{E}_{big} - \vec{E}_{small} = \frac{Q}{4\pi R^3 \varepsilon_0}(\vec{r}_1 - \vec{r}_2) = \frac{Q\vec{a}}{4\pi R^3 \varepsilon_0}$, $\vec{a}$ is not constant

$\because \vec{E}_{inside} \neq 0$, $\vec{E} = \nabla V$

$\therefore$ V is not constant

$\therefore$ the electric potential does not have same value every point in cavity

Space for extra work.

Space for extra work.