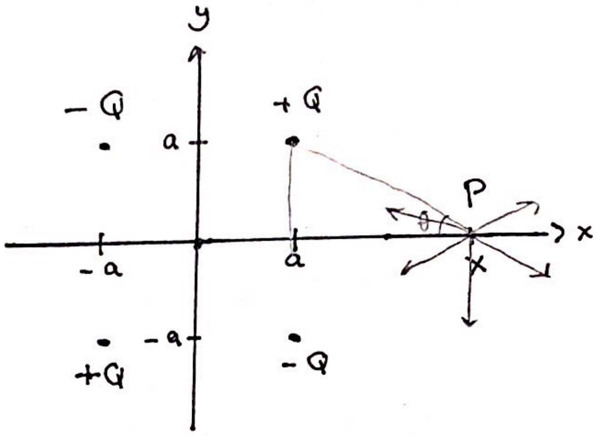
(25 Pts)

1. Two positive and two negative point charges with magnitude Q are located at the following (x, y) coordinates as shown:

(1) +Q (a, a) (3) - Q (- a, a)

(2) - Q (a, - a) (4) +Q (- a, - a)



(15) a. For the pair of charges at $x = a, y = \pm a$, find the direction of the electric field that the charges produce at Point P (x, y = 0), and show that the magnitude of the electric field $|\underline{E}|$ is

$$|\underline{E}| = \frac{2Qa}{4\pi\epsilon_0} \frac{1}{[(x-a)^2+a^2]^{3/2}}$$

(5) b. Find the total electric field at Point P that is produced by the four charges.

(5) c. Show that if x >> a, the magnitude of the total electric field is approximately given by

$$|\underline{E}| \approx \frac{12Qa^2}{4\pi\epsilon_0 x^4}$$

What type of electric field does this represent?

a. Due to symmetry, the electric field points in the negative y direction.

$$E_y = \frac{Q}{4\pi\varepsilon_0((x-a)^2+a^2)} \sin\theta + \frac{-Q}{4\pi\varepsilon_0((x-a)^2+a^2)} \sin(-\theta)$$

$\sin(-\theta) = -\sin\theta$

$$= \frac{2Q}{4\pi\varepsilon_0((x-a)^2+a^2)} \cdot \frac{a}{\sqrt{(x-a)^2+a^2}} = \frac{2Qa}{4\pi\varepsilon_0} \frac{1}{[(x-a)^2+a^2]^{\frac{3}{2}}}$$

15

b. Due to the other two charges,

$$E_y = \frac{2Qa}{4\pi\varepsilon_0} \frac{1}{[(x+a)^2+a^2]^{\frac{3}{2}}} \quad \text{(positive y-direction)}$$

5

So

$$\|\vec{E}\| = \frac{2Qa}{4\pi\varepsilon_0}\left(\frac{1}{[(x+a)^2+a^2]^{\frac{3}{2}}} - \frac{1}{[(x-a)^2+a^2]^{\frac{3}{2}}}\right)$$

3

c. $$\|\vec{E}\| = \frac{2Qa}{4\pi\varepsilon_0}\left(\frac{1}{a^3\left[\frac{(x+a)^2}{a^2}+1\right]^{\frac{3}{2}}} - \frac{1}{a^3\left[\frac{(x-a)^2}{a^2}+1\right]^{\frac{3}{2}}}\right) \approx \frac{2Q}{4\pi\varepsilon_0 a^2}\left(1 - \frac{3}{2}\frac{(x+a^2)}{a^2} - 1 + \frac{3}{2}\frac{(x-a)^2}{a^2}\right)$$

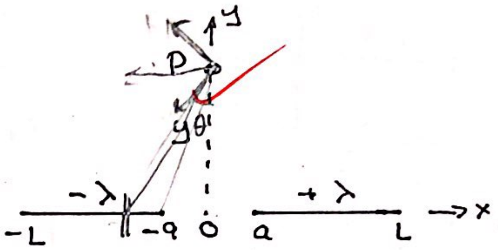
$3x^2 - 6ax + 3a^2 + 3x^2 + 6ax + 3a^2$

$$= \frac{2Q}{4\pi\varepsilon_0 a^2}\left(\frac{3(x-a)^2+3(x+a)^2}{2a^2}\right) = \frac{2Q}{4\pi\varepsilon_0 a^2}\left(\frac{6(x^2+a^2)}{2a}\right) = \frac{12Q(x^2+a^2)}{8\pi\varepsilon_0 a^3}$$

(20 Pts)

2. A thin rod with a uniform charge per unit length (dQ/dx) of $+\lambda$ extends along the x-axis from x = a to x = L. A second thin rod with dQ/dx $= -\lambda$ extends along the x-axis from x = - a to x = - L.

(15) a. Find the direction of the electric field at the Point P (x = 0, y) that is produced by the charged rods, and show that the magnitude of the electric field is given by



$$|\underline{E}| = \frac{2\lambda}{4\pi\epsilon_0}\left[\frac{1}{(y^2+a^2)^{\frac{1}{2}}} - \frac{1}{(y^2+L^2)^{\frac{1}{2}}}\right]$$

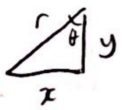
(5) b. If y >> a , and y >> L, show that the magnitude of the electric field is approximately given by

$$|\underline{E}| \approx \frac{\lambda(L^2-a^2)}{4\pi\epsilon_0 y^3}$$

What type of field does $|\underline{E}|$ represent? Note that $\lambda(L^2 - a^2) = 2\lambda(\frac{L+a}{2})(L-a)$; what quantity does this expression represent?

a) Due to symmetry, the electric field points in the negative x-direction. Also, we can calculate the x-contribution of one rod and just double it.

$$dE_x = \frac{dq}{4\pi\varepsilon_0 r^2}\sin\theta$$



$r = \frac{y}{\cos\theta}$ $\qquad dq = \lambda dx$

$x = y\tan\theta,\quad dx = \frac{y}{\cos^2\theta}d\theta$

$$= \frac{\lambda dx}{4\pi\varepsilon_0 r^2}\sin\theta$$

$$\int dE_x = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{y}{\cos^2\theta}\cdot\frac{\cos^2\theta}{y^2}\cdot\sin\theta\, d\theta$$

$$\|\vec{E}_1\| = \frac{\lambda}{4\pi\varepsilon_0 y}\int \sin\theta\, d\theta = \frac{\lambda}{4\pi\varepsilon_0 y}\Big[-\cos\theta\Big]_{\theta_1}^{\theta_2} = \frac{\lambda}{4\pi\varepsilon_0 y}\left[\frac{y}{\sqrt{y^2+a^2}} - \frac{y}{\sqrt{y^2+L^2}}\right]$$

$$\|\vec{E}\| = 2\|\vec{E}_1\| = \frac{2\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{y^2+a^2}} - \frac{1}{\sqrt{y^2+L^2}}\right]$$

b on back

b $\|\vec{E}\| = \frac{2\lambda}{4\pi\varepsilon_0}\left[\frac{1}{y\sqrt{1+\frac{a^2}{y^2}}} - \frac{1}{y\sqrt{1+\frac{L^2}{y^2}}}\right]$

$$= \frac{2\lambda}{4\pi\varepsilon_0 y}\left[\frac{1}{\sqrt{1+\frac{a^2}{y^2}}} - \frac{1}{y\sqrt{1+\frac{L^2}{y^2}}}\right]$$

$$\approx \frac{2\lambda}{4\pi\varepsilon_0 y}\left[1 - \frac{a^2}{2y^2} - 1 + \frac{L^2}{2y^2}\right]$$

$$= \frac{2\lambda}{4\pi\varepsilon_0 y}\left(\frac{L^2-a^2}{2y^2}\right)$$

$$= \frac{\lambda(L^2-a^2)}{4\pi\varepsilon_0 y^3}$$

This is the field for a quadrupole X

$$2\lambda\frac{(L+a)}{2} = Q, \quad (L-a) = d \qquad -2$$

so $\lambda(L^2-a^2) = Qd = \|\vec{p}\|$

(28 Pts)

3. A spherical insulator with radius a carries a total charge of + Q that is uniformly distributed throughout its volume ($4\pi a^3/3$). The insulator sphere is surrounded by a concentric conducting spherical shell with an inner radius b (> a) and an outer radius c (> b) as shown. A total charge of - 2Q resides on the conducting shell.

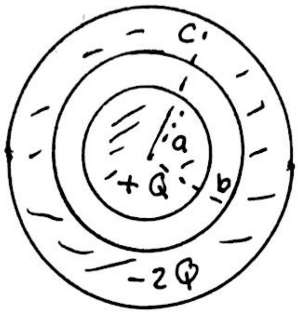
(6) a. Sketch the electric field lines for this system of charges, and explain the distribution of the -2Q charge on the conducting shell.

(12) b. Use Gauss's Law to find the electric field in the following regions:

(i) 0 < r < a (iii) b < r < c

(ii) a < r < b (iv) c < r

(10) c. Take the zero of the electric potential V (r) to be at r = 0. Show that V (r) at the surface r = c of the conducting shell is given by



$$V(r = c) = -\frac{Q}{4\pi\epsilon_0}\left[\frac{3}{2a} - \frac{1}{b}\right]$$

a. The inner surface of the conductor has charge $-Q$, and the outer also has charge $-Q$.

3

b. Because of symmetry, $\vec{E}$ has only a radial component E_r

i) $\Phi_E = EA = E_r 4\pi r^2 = \dfrac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0}$

$E_r = \dfrac{r\rho}{3\varepsilon_0} = \dfrac{Qr}{4\pi\varepsilon_0 a^3}$

$\rho = \dfrac{Q}{\frac{4}{3}\pi a^3}$

ii) $\Phi_E = EA = E_r 4\pi r^2 = \dfrac{Q}{\varepsilon_0}$

$E_r = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

iii) Inside an isolated conductor, $\vec{E} = 0$

(q enclosed = 0)

$E_r = 0$

iv) $\Phi_E = EA = E_r 4\pi r^2 = \dfrac{-Q}{\varepsilon_0}$

$E_r = -\dfrac{Q}{4\pi\varepsilon_0 r^2}$

12

(on back

c) $V(r) = -\int_0^a \frac{Qr}{4\pi\varepsilon_0 a^3}\,dr - \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}\,dr - \int_b^c 0 - \int_c^\infty$

6

$V_{\text{surface of inner sphere}} = -\frac{3}{2}\frac{Q}{4\pi\varepsilon_0 a}$ ($V_{center} = \frac{3}{2}V_{surface}$ for an insulated sphere)

why?

$V = \int_\infty^r -\frac{Q}{4\pi\varepsilon_0 r^2} + \int_r^b$

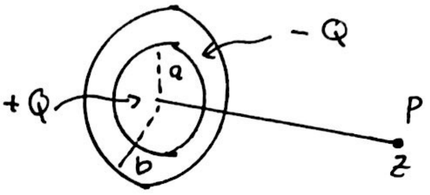
(27 Pts)

4. A thin circular disk carries a charge of $+Q$ that is uniformly distributed over the radial region $0 < R < a$ (area = πa^2), and a charge of $-Q$ that is uniformly distributed over the annular radial range $a < R < b$ (area = $\pi(b^2 - a^2)$) as shown. Consider a Point P that is at a distance z along the axis of symmetry of the disk.

(15) a. Show that the electric potential at P which is produced by the <u>positively charged inner disk</u> is

26

$$V_+ = \frac{Q}{2\pi\epsilon_0 a^2}\left[(z^2 + a^2)^{\frac{1}{2}} - z\right]$$



(7) b. Show that the electric potential at P which is produced by the <u>negatively charged annulus</u> is

$$V_- = -\frac{Q}{2\pi\epsilon_0(b^2 - a^2)}\left[(z^2 + b^2)^{\frac{1}{2}} - (z^2 + a^2)^{\frac{1}{2}}\right]$$

(5) c. For z >> a, and z >> b, show that the total electric potential $V = V_+ + V_-$ at P is approximately given by

$$V \approx \frac{Qb^2}{16\pi\epsilon_0 z^3}$$

Why does this charge distribution not produce a dipole contribution to the potential?

a Consider the contribution due to an infinitessimally thin ring with radius r $(0<r<a)$

$$dV_{ring} = \frac{dq}{4\pi\epsilon_0\sqrt{r^2+z^2}} \qquad V_{ring} = \frac{Q}{4\pi\epsilon_0\sqrt{r^2+z^2}}$$

$A = \pi r^2 \qquad dq = \sigma dA$

$dA = 2\pi r\,dr \qquad = \sigma 2\pi r\,dr$

It follows that:

$$\int dV_+ = \int \frac{dq}{4\pi\epsilon_0\sqrt{r^2+z^2}} = \frac{\sigma}{2\epsilon_0}\int\frac{r\,dr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+z^2}\right]_0^a \qquad \sigma = \frac{Q}{\pi a^2}$$

$$V_+ = \frac{Q}{2\pi\epsilon_0 a^2}\left[\sqrt{z^2+a^2} - z\right]$$

b. Again, consider contribution to infinitessimally thin ring:

Area $= \pi(b^2-a^2)$

$$\int dV_- = \int\frac{dq}{4\pi\epsilon_0\sqrt{r^2+z^2}} = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+z^2}\right]_a^b = \frac{\sigma}{2\epsilon_0}\left[\sqrt{z^2+b^2} - \sqrt{z^2+a^2}\right] \qquad \sigma = \frac{-Q}{\pi(b^2-a^2)}$$

$$V_- = -\frac{Q}{2\pi\epsilon_0(b^2-a^2)}\left[\sqrt{z^2+b^2} - \sqrt{z^2+a^2}\right]$$

c on back

C. $V = V_+ + V_- = \frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{a^2}\left(\sqrt{z^2+a^2} - z\right) - \frac{1}{b^2-a^2}\left(\sqrt{z^2+b^2} - \sqrt{z^2+a^2}\right)\right]$

$$= \frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{a^2}\left(z\sqrt{1+\frac{a^2}{z^2}} - z\right) - \frac{1}{b^2-a^2}\left(z\sqrt{1+\frac{b^2}{z^2}} - z\sqrt{1+\frac{a^2}{z^2}}\right)\right.$$

$$= \frac{Qz}{2\pi\varepsilon_0}\left[\frac{1}{a^2}\left(\sqrt{1+\frac{a^2}{z^2}} - 1\right) - \frac{1}{b^2-a^2}\left(\sqrt{1+\frac{b^2}{z^2}} - \sqrt{1+\frac{a^2}{z^2}}\right)\right]$$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$V \approx \frac{Qz}{2\pi\varepsilon_0}\left[\frac{1}{a^2}\left(1+\frac{a^2}{2z^2} - \frac{a^4}{8z^4} - 1\right) - \frac{1}{b^2-a^2}\left(1+\frac{b^2}{2z^2} - \frac{b^4}{8z^4} - 1 - \frac{a^2}{2z^2} + \frac{a^4}{8z^4}\right)\right.$$

$$= \frac{Qz}{2\pi\varepsilon_0}\left[\frac{1}{2z^2} - \frac{a^4}{8z^4} - \frac{1}{\cancel{b^2-a^2}}\left(\frac{\cancel{b^2-a^2}}{2z^2} + \frac{a^4 - b^4}{8z^4}\right)\right]$$

$$= \frac{Qz}{2\pi\varepsilon_0}\left[\frac{1}{2z^2} - \frac{a^4}{8z^4} - \frac{1}{\cancel{b^2-a^2}}\left(\frac{\cancel{(a^2-b^2)}(a^2+b^2)}{8z^4}\right)\right]$$

$$= \frac{Qz}{2\pi\varepsilon_0}\left[\frac{1}{2z^2} - \frac{a^4}{8z^4} + \frac{a^2+b^2}{8z^4}\right]$$

$$= \frac{Qz}{2\pi\varepsilon_0}\left[\frac{1}{2z^2} + \frac{a^2+b^2-a^4}{8z^4}\right]$$

4