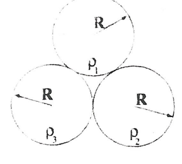
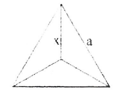
$q_i = \frac{4}{3}\pi R^3 \rho_i$

$r_{ij} = 2R$

$a = 2R$
$x = \frac{2}{\sqrt{3}} R$




$a^2 = 3x^2$

1: Three spheres of identical radius $R$ (but different, uniform charge densities $\rho_1$, $\rho_2$ and $\rho_3$) are arranged so that they touch one another as shown in the diagram above. [The diagram at the right is a hint for later].

- 1a) (10 points) How much work will it take to assemble these spheres into the arrangement shown? Assume the spheres themselves have already been assembled - that is, neglect the self-energy of each sphere).

$$W_{ext} = \frac{1}{4\pi\epsilon_0} \sum \frac{q_i q_j}{r_{ij}}$$

$$W_{ext} = \frac{1}{4\pi\epsilon_0}\left(\frac{4}{3}\pi R^3\right)^2\left(\frac{1}{2R}\right)\left[\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3\right]$$

$$\boxed{W_{ext} = \frac{2\pi R^5}{9\epsilon_0}\left(\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3\right)}$$

- 1b) (10 points) What is the electric potential at the center of the arrangement?

$$V_i = \frac{1}{4\pi\epsilon_0}\frac{q_i}{x}$$

$$V_i = \frac{1}{4\pi\epsilon_0}\frac{4}{3}\pi R^3 \rho_i \frac{\sqrt{3}}{2R}$$

$$V_i = \frac{R^2}{2\sqrt{3}\epsilon_0}\rho_i$$

$$V = \sum V_i \Rightarrow \boxed{V = \frac{R^2}{2\sqrt{3}\epsilon_0}(\rho_1 + \rho_2 + \rho_3)}$$

- 1c) (10 points) What is the electric field at the the center of the arrangement?

$$\vec{E}_i = \frac{q_i}{4\pi\epsilon_0 x^2}\hat{r}_i$$

$$\vec{E}_i = \frac{1}{4\pi\epsilon_0}\frac{4}{3}\pi R^3 \rho_i \frac{3}{4R^2}\hat{r}_i$$

$$\vec{E}_i = \frac{R}{4\epsilon_0}\rho_i \hat{r}_i$$

$$\vec{E}_1 = \frac{R}{4\epsilon_0}\left[\qquad -\rho_1 \hat{j}\right]$$

$$\vec{E}_2 = \frac{R}{4\epsilon_0}\left[-\rho_2 \cos\theta\,\hat{\imath} + \rho_2 \sin\theta\,\hat{j}\right]$$

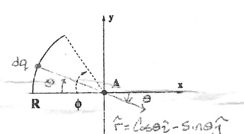
$$\vec{E}_3 = \frac{R}{4\epsilon_0}\left[\rho_3 \cos\theta\,\hat{\imath} + \rho_3 \sin\theta\,\hat{j}\right]$$

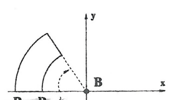
$$\vec{E} = \sum \vec{E}_i$$

$$\vec{E} = \frac{R}{4\epsilon_0}\left[(\rho_3 - \rho_2)\cos\theta\,\hat{\imath} + \left[(\rho_3 + \rho_2)\sin\theta - \rho_1\right]\hat{j}\right]$$

$$\theta = 30^\circ \Rightarrow \cos(30^\circ) = \sqrt{3}/2 \qquad \sin(30^\circ) = 1/2$$

$$\boxed{\vec{E} = \frac{R}{8\epsilon_0}\left[\sqrt{3}(\rho_3 - \rho_2)\,\hat{\imath} + (\rho_3 + \rho_2 - 2\rho_1)\,\hat{j}\right]}$$




$dq = \lambda ds$
$dq = \frac{q}{\phi}d\theta$

- 2a) (10 points) A thin nonconducting rod that carries an electric charge $q$ (uniformly distributed) is bent to form a circular arc of radius $R$ that subtends an angle $\phi$ as shown in the diagram on the left. Find the electric field (vector) at point $A$ (located at the center of curvature of the arc).

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 r^2}\hat{r}$$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0 R^2}\frac{q\,d\theta}{\phi}(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{j})$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0 \phi R^2}\int_0^\phi d\theta(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{j})$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 \phi R^2}\left[\sin\phi\,\hat{\imath} - (1 - \cos\phi)\,\hat{j}\right]}$$

Alternately,
$\sin\phi = 2\sin\phi/2 \cos\phi/2$
$1 - \cos\phi = 2\sin^2\phi/2$

$$\boxed{\vec{E} = \frac{Q\sin\phi/2}{2\pi\epsilon_0 \phi R^2}\left[\cos\phi/2\,\hat{\imath} - \sin\phi/2\,\hat{j}\right]}$$

- 2b) (10 points) Now consider the diagram shown on the right. Charge is spread over a wedge defined by the angle $\phi$ between the radial distances $R_1$ and $R_2$ (as shown) with an area charge density

$$\sigma(r) = \frac{1Q}{\phi(R_2^4 - R_1^4)}r^2$$

where $r$ is the radial distance from the $B$ (located at the center of curvature of the defining arcs). Find the electric field (vector) at $B$.

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 \phi r^2}\left[\sin\phi\,\hat{\imath} - (1 - \cos\phi)\,\hat{j}\right]$$

$$dq = \sigma dA = \frac{4Q}{\phi(R_2^4 - R_1^4)}r^2 \cdot \phi r dr$$

$$dq = \frac{4Q r^3 dr}{(R_2^4 - R_1^4)}$$

$$\int d\vec{E} = \frac{Q\left[\sin\phi\,\hat{\imath} - (1 - \cos\phi)\,\hat{j}\right]}{\pi\epsilon_0 \phi (R_2^4 - R_1^4)}\int_{R_1}^{R_2} dr\, r$$

$$\boxed{\vec{E} = \frac{Q}{2\pi\epsilon_0 \phi (R_1^2 + R_2^2)}\left[\sin\phi\,\hat{\imath} - (1 - \cos\phi)\,\hat{j}\right]}$$

Either form is fine

$$\boxed{\vec{E} = \frac{Q\sin\phi/2}{\pi\epsilon_0 \phi (R_1^2 + R_2^2)}\left[\cos\phi/2\,\hat{\imath} - \sin\phi/2\,\hat{j}\right]}$$

- 2b) *(continued...)*

- 2c) (10 points) Find the electric potential produced by the wedge at point $B$ relative to a point infinitely-distant from the wedge.

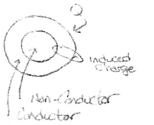
All the points in a thin arc are equidistant to B, so this is easier than it looks :)

$$dV = \frac{dq}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0 r}\frac{4Q r^3 dr}{(R_2^4 - R_1^4)}$$

$$\int dV = \frac{Q}{\pi\epsilon_0 (R_2^4 - R_1^4)}\int_{R_1}^{R_2} dr\, r^2$$

$$\boxed{V = \frac{Q\,(R_2^3 - R_1^3)}{3\pi\epsilon_0 (R_2^4 - R_1^4)}}$$

3) A spherical charge distribution of radius $R$ carries a volume charge density




$$\rho(r) = \rho_0\,(1 - \frac{r^3}{R^3})$$

It is surrounded by a concentric spherical conducting shell that extends from $r = R$ to $r = 2R$ and carries an excess charge $Q$.

- 3a) (10 points) Find the charge inside a concentric sphere of radius $r$, for all values of $r$. Also, find the surface charge densities on the inner and outer surfaces of the conducting shell.

Spherical dist: $dq = \rho\,dv = \rho\,4\pi r^2 dr$

$(r<R)$ $dq = \rho\,dV = \rho_0\,(1 - r^3/R^3)\,4\pi r^2$

$q_{in}(r) = \int dq = 4\pi\rho_0 \int_0^r (r^2 - r^5/R^3)\,dr$

$q_{in}(r) = 4\pi\rho_0\,\frac{1}{6}\left[2r^3 - \frac{r^6}{R^3}\right]$

$q_{in}(r) = \frac{2}{3}\pi\rho_0 r^3\,(2 - r^3/R^3)$

$(R<r<2R)$ $q_{in}(r) = 0$

$(2R<r)$ $q_{in}(r) = q_{in}(R) + Q$

$q_{in}(r) = \frac{2}{3}\pi\rho_0 R^3 + Q$

$\sigma_{in}(R) = \frac{-q_{in}(R)}{4\pi R^2} = -\frac{\rho_0 R}{6}$

$\sigma_{in}(2R) = \frac{q_{in}(R) + Q}{4\pi(2R)^2} = \frac{\frac{2\pi\rho_0 R^3}{3} + Q}{16\pi R^2}$

$$\boxed{\begin{array}{ll} q_{in}(r) = \frac{2\pi}{3}\rho_0 r^3(2 - r^3/R^3) & (r<R) \\ q_{in}(r) = 0 & (R<r<2R) \\ q_{in}(r) = \frac{2\pi}{3}\rho_0 R^3 + Q & (2R<r) \\ \sigma_{in}(R) = -\frac{\rho_0 R}{6} & \\ \sigma_{in}(2R) = \frac{\rho_0 R}{24} + \frac{Q}{16\pi R^2} & \end{array}}$$

- 3b) (10 points) Find the electric field as a function of the radial distance from the center of the charge distribution ($r$) for all values of $r$.

Spherical Symmetry: $\vec{E}(r) = \frac{q_{in}(r)}{4\pi\epsilon_0 r^2}\hat{r}$

$$\boxed{\begin{array}{ll} (r<R) & \vec{E} = \frac{\rho_0}{6\epsilon_0} r\,(2 - r^3/R^3)\,\hat{r} \\ (R<r<2R) & \vec{E} = 0 \\ (2R<r) & \vec{E} = \frac{1}{4\pi\epsilon_0 r^2}\left(\frac{2\pi\rho_0 R^3}{3} + Q\right)\hat{r} \end{array}}$$

- 3c) (10 points) If the electric potential within the conductor is given as $V_0$, find the potential as a function of the radial distance from the center of the charge distribution ($r$) for all values of $r$.

Spherical Symmetry – $\vec{E}\cdot d\vec{r} = E_r dr$

$\Delta V(\vec{r}_{ref}, \vec{r}) = -\int_{r_{ref}}^{r} \vec{E}\cdot d\vec{r}$

$V(r) = V(r_{ref}) - \int_{r_{ref}}^{r} E_r\,dr$

$V(r) = V_0 - \int_{r_{ref}}^{r} E_r\,dr$

$(r<R)$ $V(r) = V_0 - \int_R^r \frac{\rho_0}{6\epsilon_0} r\,(2 - r^3/R^3)\,dr$

$V(r) = V_0 - \frac{\rho_0}{6\epsilon_0}\left[r^2 - \frac{1}{5} r^5/R^3\right]_R^r$

$V(r) = V_0 - \frac{\rho_0 R^2}{30\epsilon_0}\left[5 r^2/R^2 - r^5/R^5 - 4\right]$

$(R<r<2R)$ $V(r) = V_0 - \int_R^r 0\,dr$

$V(r) = V_0$

$(2R<r)$ $V(r) = V_0 - \left(\frac{2\pi\rho_0 R^3}{3} + Q\right)\frac{1}{4\pi\epsilon_0}\int_R^r \frac{dr}{r^2}$

$V(r) = V_0 + \frac{1}{4\pi\epsilon_0}\left(\frac{2\pi\rho_0 R^3}{3} + Q\right)\left(\frac{1}{r} - \frac{1}{R}\right)$

$$\boxed{\begin{array}{ll} V(r) = V_0 + \frac{\rho_0 R^2}{30\epsilon_0}\left[4 - 5\frac{r^2}{R^2} + \frac{r^5}{R^5}\right] & (r<R) \\ V(r) = V_0 & (R<r<2R) \\ V(r) = V_0 + \frac{1}{4\pi\epsilon_0}\left[\frac{2\pi\rho_0 R^3}{3} + Q\right]\left(\frac{1}{r} - \frac{1}{R}\right) & (2R<r) \end{array}}$$