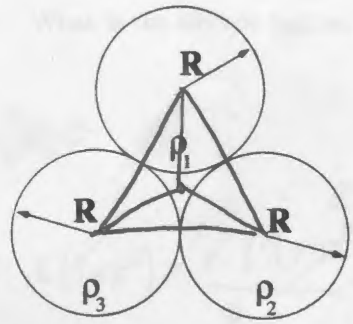
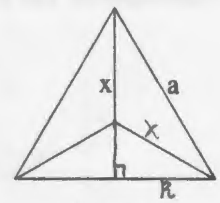




$a^2 = 3x^2$

$x^2 = \frac{a^2}{3}$ $x = \sqrt{\frac{a^2}{3}}$ $a = 2R$

1) Three spheres of identical radius R (but different, uniform charge densities ρ_1, ρ_2 and ρ_3) are arranged so that they touch one another as shown in the diagram above. [The diagram at the right is a hint for later].

+10

- 1a) (10 points) How much work will it take to assemble these spheres into the arrangement shown? [Assume the spheres themselves have already been assembled - that is, neglect the self-energy of each sphere].

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r} \Rightarrow W_{ext} = U_{\rho_2} + U_{\rho_3}$$

$$Q_i = \rho_i \cdot \frac{4}{3}\pi R^3$$

$$= \frac{\left(\frac{4}{3}\pi R^3\right)^2 \rho_1 \rho_2}{4\pi\epsilon_0 (2R)} + \frac{\left(\frac{4}{3}\pi R^3\right)^2 \rho_1 \rho_3}{4\pi\epsilon_0 (2R)} + \frac{\left(\frac{4}{3}\pi R^3\right)^2 \rho_2 \rho_3}{4\pi\epsilon_0 (2R)}$$

$$2\,\frac{\cancel{16}}{\cancel{9}}\frac{\cancel{\pi^2} R^6}{\pi}$$

$$= \frac{\left(\frac{4}{3}\pi R^3\right)^2}{4\pi\epsilon_0 (2R)}\left[\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3\right] = \boxed{\frac{2\pi R^5}{9\epsilon_0}\left[\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3\right]}$$

$$\cancel{8\pi}\epsilon_0 R$$

+9

- 1b) (10 points) What is the electric potential at the center of the arrangement?

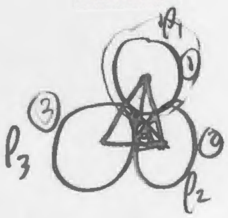
$$V = \sum \frac{Q_i}{4\pi\epsilon_0 r_i} = \sum \frac{\rho_i \frac{\cancel{4}}{3}\cancel{\pi} R^3}{\cancel{4\pi}\epsilon_0 \left(\sqrt{\frac{4R^2}{3}}\right)} = \sum \frac{\rho_i R^3}{\frac{(2R)}{\sqrt{3}}(3\epsilon_0)} = \frac{\rho_i R^2 \sqrt{3}}{2\epsilon_0 \sqrt{3}}$$

$$Q_1 = \rho_1 \frac{4}{3}\pi r^3$$

$$\Rightarrow \boxed{V_{total} = \frac{R^2\sqrt{3}}{2\epsilon_0}(\rho_1 + \rho_2 + \rho_3)}$$

+4

• 1c) (10 points) **What is the electric field at the the center of the arrangement?**

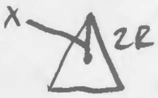


$E_{tot} = \sum E_i \hat{r}$

NO $\frac{4R^3}{3}$

R

① : $E(4\pi R^2) = \dfrac{\rho_1 (\frac{4}{3}\pi R^3)}{\varepsilon_0} \Rightarrow \vec{E} = \dfrac{-\rho_1 R}{3\varepsilon_0}\hat{j}$



$3x^2 = 4R^2$

$x = \dfrac{2R}{\sqrt{3}}$

② : $\vec{E} = \dfrac{-\rho_2 R}{3\varepsilon_0}\left(-\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$

③ $\vec{E} = \dfrac{\rho_3 R}{3\varepsilon_0}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$

$\sqrt{x^2 - R^2}$

$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$

$= \left[\dfrac{-\rho_1 R}{3\varepsilon_0} + \dfrac{\rho_2 R}{3\varepsilon_0}\sin\theta + \dfrac{\rho_3 R}{3\varepsilon_0}\sin\theta\right]\hat{j}$

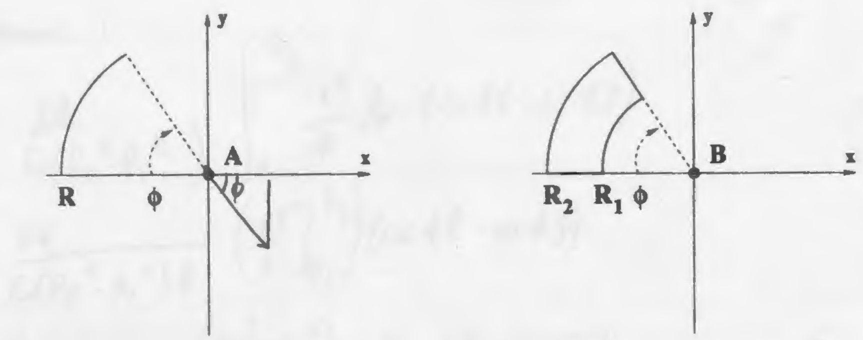
$= \sqrt{\dfrac{4R^2}{3} - R^2}$

$= \sqrt{\dfrac{R^2}{3}}$

$$\boxed{= \left(\dfrac{-\rho_1 R}{3\varepsilon_0} + \dfrac{\rho_2 R}{6\varepsilon_0} + \dfrac{\rho_3 R}{6\varepsilon_0}\right)\hat{j}}$$

$= \dfrac{R}{\sqrt{3}}$

$\sin\theta = \dfrac{\frac{R}{\sqrt{3}}}{\frac{2R}{\sqrt{3}}} = \dfrac{1}{2}$



$\frac{q}{L} = \lambda = \lambda$

- 2a) (10 points) A thin nonconducting rod that carries an electric charge q (uniformly distributed) is bent to form a circular arc of radius R that subtends an angle ϕ as shown in the diagram on the left. Find the electric field (vector) at point A (located at the center of curvature of the arc).

$$\vec{E} = \int \vec{dE} = \int \frac{dq}{4\pi\varepsilon_0 r^2} = \int \frac{dq}{4\pi\varepsilon_0 R^2}\hat{r} = \frac{\lambda}{4\pi\varepsilon_0} \int_0^{\phi} \frac{1}{R^2} \cdot R\, d\phi\, \hat{r} = \frac{\lambda}{4\pi\varepsilon_0 R} \int_0^{\phi} d\phi\, \hat{r}$$

$r = R$

$dq = \lambda ds = \lambda R d\phi$

$$= \frac{\lambda}{4\pi\varepsilon_0} (\phi)\hat{r}$$

$s = 2\pi R \cdot \frac{\phi}{2\pi}$

$= R\phi$

$\lambda = \frac{q}{s} = \frac{q}{R\phi}$

+5

$$= \frac{q}{4\pi\varepsilon_0 (R\cancel{\phi})} \cdot \cancel{\phi} \left[\cos\phi\,\hat{j} - \sin\phi\,\hat{i}\right]$$

$$= \boxed{\frac{q}{4\pi\varepsilon_0 R}\left[\cos\phi\,\hat{j} - \sin\phi\,\hat{i}\right]}$$

- 2b) (10 points) Now consider the diagram shown on the right. Charge is spread over a wedge defined by the angle ϕ between the radial distances R_1 and R_2 (as shown) with an area charge density

$$\sigma(r) = \frac{4Q}{\phi(R_2^4 - R_1^4)} r^2$$

where r is the radial distance from the B (located at the center of curvature of the defining arcs). Find the electric field (vector) at B.

$$\vec{E} = \int \vec{dE} = \int_{R_1}^{R_2} \vec{dE}_{arc}\,\hat{r} = \int_{R_1}^{R_2} \frac{dq}{4\pi\varepsilon_0 r} (\cos\phi\,\hat{i} - \sin\phi\,\hat{j}) = \int_{R_1}^{R_2} \frac{\sigma(r)}{2\varepsilon_0} (\cos\phi\,\hat{i} - \sin\phi\,\hat{j})\, dr$$

$dq = \sigma(r)\, dA$

$= \sigma(r) \cdot 2\pi r dr$

$$= \frac{1}{2\varepsilon_0} \int_{R_1}^{R_2} \frac{4Q}{\phi(R_2^4 - R_1^4)} r^2 dr\, (\cos\phi\,\hat{i} - \sin\phi\,\hat{j})$$

+7

$$= \frac{2Q}{\varepsilon_0 (R_2^4 - R_1^4)} \int_{R_1}^{R_2} \frac{r^2}{\phi} dr\, (\cos\phi\,\hat{i} - \sin\phi\,\hat{j})$$

(next page)

- 2b) *(continued...)*

$$\vec{E} = \frac{2Q}{\varepsilon_0(R_2^4 - R_1^4)} \int_{R_1}^{R_2} \frac{r^2}{\phi}\, dr \,(\cos\phi\,\hat{i} - \sin\phi\,\hat{j})$$

$$= \frac{2Q}{\varepsilon_0(R_2^4 - R_1^4)\phi} \left(\frac{r^3}{3}\right]_{R_1}^{R_2} (\cos\phi\,\hat{i} - \sin\phi\,\hat{j})$$

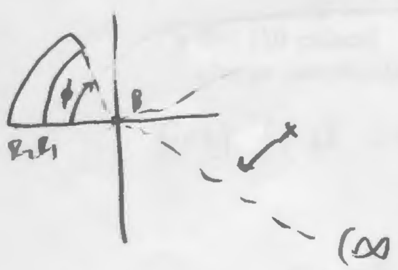
$R_2^3 - R_1^3 = (R_2 - R_1)(R_2^2 + 2R_2R_1 + R_1^2)$

$R_2^4 - R_1^4 = (R_2^2 - R_1^2)(R_2^2 + R_1^2)$

$$= \frac{2Q}{\varepsilon_0(R_2^4 - R_1^4)} \frac{(R_2^3 - R_1^3)}{3} [\cos\phi\,\hat{i} - \sin\phi\,\hat{j}]$$

$$= \boxed{\frac{2Q\,(R_2^2 + 2R_2R_1 + R_1^2)}{3\varepsilon_0(R_2 + R_1)(R_2^2 + R_1^2)} (\cos\phi\,\hat{i} - \sin\phi\,\hat{j})}$$

- 2c) (10 points) **Find the electric potential produced by the wedge at point B relative to a point infinitely-distant from the wedge.**



$$V(B) = V_r - V_{r_{ref}} = V(B) - V(\infty)$$

$$= -\int_{\infty}^{B} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{B} E(r)\,dr = -\int_{\infty}^{0} \frac{2Q(R_2^2 + 2R_2R_1 + R_1^2)}{\varepsilon_0(R_2 + R_1)(R_2^2 + R_1^2)}$$

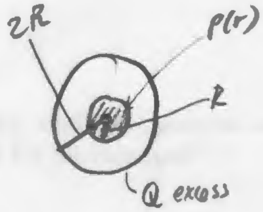
$dE_r = dE_{wedge}$

$$E(r) = \frac{2Q\left[(R_2 + x)^3 - (R_1 + x)^3\right]}{\varepsilon_0\left[(R_2 + x)^4 - (R_1 + x)^4\right]} [\cos\phi\,\hat{i} - \sin\phi\,\hat{j}]$$

+1

$=$

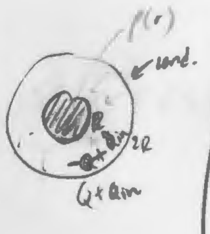
3) A spherical charge distribution of radius R carries a volume charge density



$$\rho(r) = \rho_0\,(1 - \frac{r^3}{R^3})$$

It is surrounded by a concentric spherical conducting shell that extends from $r = R$ to $r = 2R$ and carries an excess charge Q.

- 3a) (10 points) Find the charge inside a concentric sphere of radius r, for all values of r. Also, find the surface charge densities on the inner and outer surfaces of the conducting shell.



$Q(r)_{enc}$

$[r<R]$ $Q_{enc}(r) = \int_0^r \rho(r)\,dV = 4\pi\rho_0 \int r^2 - \frac{r^5}{R^3}\,dr$

$= \boxed{4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^6}{6R^3}\right)}$ 2

$[R<r<2R]$ $Q_{enc}(r) = \int_0^R \rho(r)\,dV = \boxed{4\pi\rho_0\left(\frac{R^3}{6}\right)}$ 0

$[R>2R]$ $Q_{enc}(r) = \boxed{4\pi\rho_0\left(\frac{R^3}{6}\right) + Q}$ 2

$\int \rho(r)\cdot 4\pi r^2 dr$

$= 4\pi\rho_0 \int_0^a r^2 - \frac{r^5}{R^3}\,dr$

$= 4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^6}{6R^3}\right)\Big|_0^a$

$Q_{tot} =$

$4\pi\rho_0\left(\frac{R^3}{3} - \frac{R^3}{6}\right)$

$= 4\pi\rho_0\left(\frac{R^3}{6}\right)$

$$\boxed{\sigma(r) \begin{cases} \sigma(r) = \dfrac{-Q + 4\pi\rho_0\left(\frac{R^3}{6}\right)}{4\pi r^2} & [\text{inside}] \\ \sigma(r) = \dfrac{Q + 4\pi\rho_0\left(\frac{R^3}{6}\right)}{4\pi r^2} & [\text{outside}] \end{cases}}$$

1

1

- 3b) (10 points) Find the electric field as a function of the radial distance from the center of the charge distribution (r) for all values of r.

$[r<R]$ $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}(r)}{\varepsilon_0} \rightarrow \vec{E}(4\pi r^2) = \frac{\int_0^r \rho(r)\cdot 4\pi r^2 dr}{\varepsilon_0} = \boxed{\frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^4}{6R^3}\right]\hat{r}}$ 1

$[R<r<2R]$ $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}(r)}{\varepsilon_0} \rightarrow \vec{E} = \frac{Q_{dist}}{\varepsilon_0(4\pi r^2)} = \frac{\rho_0}{3}\left[\frac{R}{3} - \frac{R}{6}\right] = \boxed{\frac{\rho_0 R}{6\varepsilon_0}\hat{r}}$ 3

0

$[r>2R]$ $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}(r)}{\varepsilon_0} \rightarrow \vec{E} = \frac{4\pi\rho_0\left(\frac{R^3}{6}\right) + Q}{\varepsilon_0(4\pi r^2)} = \boxed{\frac{\rho_0 R^3 + 6Q}{6\varepsilon_0 r^2}\hat{r}}$

7

- 3c) (10 points) **If the electric potential within the conductor is given as V_0, find the potential as a function of the radial distance from the center of the charge distribution (r) for all values of r.**

$$[r>2R] \quad V = -\oint_{\infty}^{r} E \cdot dr = -\int_{\infty}^{r} \frac{\rho_0 R^3 + 6Q}{6\varepsilon_0 r^2}\, dr$$

$$[R < r < 2R] \quad V = -\int_{\infty}^{2R} \frac{\rho_0 R^3 + 6Q}{6\varepsilon_0 r^2}\, dr + \int_{2R}^{r} \frac{\rho_0 R}{6\varepsilon_0}\, dr$$