1) It's late at night and you're busy trying to put in those extra last-minute minutes that you're absolutely certain will make all the difference in the world on tomorrow's exam. You hear buzzing. Maybe it's all that caffeine, maybe you're hallucinating, maybe it's the fly that just flew past the tip of your nose. On approach, you observe the pitch coming from the fly's wings has a frequency f_1; as it recedes, you observe that the pitch has a new frequency, f_2. The air in the room is still and under the current barometric and thermal conditions the speed of sound in air in the room is given by v_{snd}.

- 1a) (10 points) How fast is the fly moving relative to the tip of your nose?

$$\frac{f_{obs}}{f_{src}} = \frac{V_{snd} - V_{obs}}{V_{snd} - V_{src}}$$

$$\frac{f_1}{f_0} = \frac{V_{snd}}{V_{snd} - |V_x|}$$

$$\frac{f_2}{f_0} = \frac{V_{snd}}{V_{snd} + |V_x|}$$

$$\frac{f_1}{f_2} = \frac{V_{snd} + |V_x|}{V_{snd} - |V_x|}$$

$$\boxed{|V_x| = V_{snd} \frac{f_1 - f_2}{f_1 + f_2}}$$

- 1b) (10 points) What frequency would you hear if the fly were to hover at the tip of your nose?

$$f_0 \left(\frac{1}{f_1} + \frac{1}{f_2} \right) = 2$$

$$\boxed{f_0 = \frac{2 f_1 f_2}{f_1 + f_2}}$$

- 1c) (10 points) The fly makes a second pass with the same speed it had before. This time, you're ready. You hum at the frequency you would hear if the fly were to hover in front of your nose. Would the fly hear a pitch of frequency f_1 as it approaches and a frequency f_2 as it recedes? Explain. You will receive points for clarity, brevity and (correct) detail.

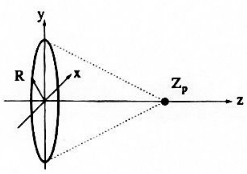
$\boxed{\text{No}}$

The Doppler effect is really made up of two distinct contributions ~

In the original problem, the source was moving and the observer was stationary. Wave crests were emitted closer or farther to/from their neighbors, and the direct effect was on wavelength ($\propto 1/f$)

In the new problem (c), the source is stationary and the observer moves through crests, encountering them more or less frequently in the process - the direct effect is on frequency

An alternate, and not quite as satisfying explanation:
The Doppler equation, as written, assumes the medium (air) is at rest. In the frame in which the observer (the fly) is at rest, the air is moving (past the fly) and we'd need to modify the equation to take this into account.



- 2a) (5 points) If the circle shown represents a uniform ring of charge Q and radius R, what is the resultant electric field at z_p?

You may write or derive this answer, you may derive it directly by $\int d\vec{E}$ or by taking the gradient of the potential...

$$\boxed{\vec{E} = \frac{Qz}{4\pi\epsilon_0 (R^2+z^2)^{3/2}} \hat{z}}$$

- 2b) (5 points) If the circle shown represents a uniform disk of charge Q and radius R, what is the resultant electric field at z_p?

This, you'll probably need to derive :) Build the disk up from infinitesimal rings -

$$d\vec{E} = \frac{dq\, z}{4\pi\epsilon_0 (r^2+z^2)^{3/2}} \hat{z}$$

$$d\vec{E} = \frac{\sigma\, dA\, z}{4\pi\epsilon_0 (r^2+z^2)^{3/2}} \hat{z}$$

$$dA = 2\pi r\, dr \qquad \sigma = \frac{Q}{\pi R^2}$$

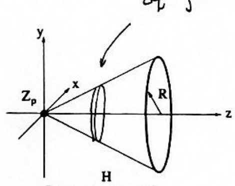
$$\int d\vec{E} = \frac{Qz\hat{z}}{4\epsilon_0 \pi R^2} \int_0^R \frac{2r\,dr}{(r^2+z^2)^{3/2}}$$

$$\vec{E} = \frac{Qz\hat{z}}{4\pi\epsilon_0 R^2} \cdot \frac{2}{\sqrt{r^2+z^2}} \Bigg|_R^0$$

$$\boxed{\vec{E} = \frac{1}{2\epsilon_0} \frac{Q}{\pi R^2} \left[1 - \frac{1}{\sqrt{1 + R^2/z^2}} \right] \hat{z}}$$

$dq = \rho\, dV = \rho \pi r^2 dz$

$\frac{r}{z} = \frac{R}{H}$



- 2c) (20 points) A charge distribution occupies the volume of a right circular cone of base-radius R and height H that is oriented so that its apex is on the origin and its longitudinal symmetry axis lies along the $+z$-axis with the base intersecting $z = H$, as shown. Assuming the charge distribution has a volume density given by

$$\rho(z) = \frac{(n+3)Q}{\pi R^2 H^{n+1}} z^n$$

find the (vector) electric field at the origin.

build the cone out of disks... note $\vec{E}$ points in the $(-\hat{z})$ direction if $Q>0$

$$d\vec{E} = \frac{-dq}{2\pi\epsilon_0 r^2} \left[1 - \frac{1}{\sqrt{1 + r^2/z^2}} \right] \hat{z}$$ ← disks of charge dq, and radius r

$$d\vec{E} = -\frac{\rho(z) \pi r^2 dz}{2\pi\epsilon_0 r^2} \left[1 - \frac{1}{\sqrt{1 + R^2/H^2}} \right] \hat{z}$$ ← $\frac{r}{z} = \frac{R}{H}$

$$\int d\vec{E} = \frac{-1}{2\epsilon_0} \left[1 - \frac{H}{\sqrt{R^2+H^2}} \right] \hat{z} \int_0^H dz\, \rho(z)$$ ← $\rho(z) = \frac{(n+3)Q}{\pi R^2 H^{n+1}} z^n$

$$\boxed{\vec{E} = -\frac{(n+3)}{(n+1)} \frac{1}{2\epsilon_0} \frac{Q}{\pi R^2} \left[1 - \frac{H}{\sqrt{R^2+H^2}} \right] \hat{z}}$$

A nonconducting sphere of radius a carries electric charge distributed with a volume charge-density $\rho(r) = \rho_0\left[1 - \frac{4}{3}\frac{r}{a}\right]$. It is surrounded by a concentric spherical conducting shell of inner-radius a and outer-radius $3a$ that carries an excess charge $-q$.

$q_{in}(r) = \int dq = \int \rho dV = \int \rho(r) 4\pi r^2 dV$ ← Spherical Symmetry → $\vec{E} = \frac{q_{in}(r)}{4\pi\epsilon_0 r^2}\hat{r}$

- **3a) (10 points)** Find the amount of charge contained in a concentric sphere of radius r, for all values of r.

$(r<a)$ $q_{in}(r) = 4\pi\rho_0 \int_0^r \left(1 - \frac{4}{3}\frac{r}{a}\right) r^2 dr$

$= \frac{4}{3}\pi\rho_0 r^3 \left(1 - r/a\right)$

$(a<r<3a)$ $q_{in}(r) = 0$ ← inside spherically symmetric conducting shell $\vec{E}=0 \Rightarrow q_{in}=0$

$(3a<r)$ $q_{in}(r) = q_{in}(a) + 0 + (-q)$ (non-conducting contribution; induced charge; excess charge)

$q_{in}(r) = -q$

$$\boxed{q_{in}(r) = \begin{cases} \frac{4}{3}\pi\rho_0 r^3 (1 - r/a) & (r<a) \\ 0 & (a<r<3a) \\ -q & (3a<r) \end{cases}}$$

- **3b) (10 points)** Find the electric field (vector) at all points inside and outside the distribution.

Spherical Symmetry: $\vec{E} = \frac{q_{in}(r)}{4\pi\epsilon_0 r^2}\hat{r}$

$$\boxed{\vec{E} = \begin{cases} \frac{\rho_0}{3\epsilon_0} r(1 - r/a)\hat{r} & (r<a) \\ 0 & (a<r<3a) \\ \frac{-q}{4\pi\epsilon_0 r^2} & (3a<r) \end{cases}}$$

- **3c) (10 points)** Find the electric potential at all points inside and outside the distribution, if the conductor is taken to be at potential V_a.

$\Delta V(a,r) = -\int_a^r \vec{E}\cdot d\vec{s}$

$V(r) - V(a) = -\int_a^r E_r dr$ ← spherical symmetry / radial field

$V(r) = V_a - \int_a^r E_r dr$ ← $V(a) \equiv V_a$

$(r<a)$ $V(r) = V_a - \int_a^r \frac{\rho_0}{3\epsilon_0} r(1 - r/a) dr$

$= V_a - \frac{\rho_0}{3\epsilon_0}\left[\frac{r^2}{2} - \frac{r^3}{3a}\right]_a^r$

$= V_a - \frac{\rho a^2}{18\epsilon_0}\left(3\frac{r^2}{a^2} - 2\frac{r^3}{a^3} - 1\right)$

$(a<r<3a)$ $V(r) = V_a - \int_a^r 0 \cdot dr = V_a$

$(3a<r)$ $V(r) = V_a - \int_a^{3a} 0 \cdot dr - \int_{3a}^r \frac{-q}{4\pi\epsilon_0 r^2} dr$

$= V_a - \frac{q}{4\pi\epsilon_0 r}\Big|_{3a}^r$

$= V_a - \frac{q}{4\pi\epsilon_0 r}\left[1 - \frac{r}{3a}\right]$

$$\boxed{V(r) = \begin{cases} V_a + \frac{\rho a^2}{18\epsilon_0}\left[1 - 3\frac{r^2}{a^2} + 2\frac{r^3}{a^3}\right] & (r<a) \\ V_a & (a<r<3a) \\ V_a - \frac{q}{12\pi\epsilon_0 r}\left[3 - \frac{r}{a}\right] & (3a<r) \end{cases}}$$