1) You're standing on the edge of a cliff. Out in the distance, at roughly the same elevation as you, you see a vulture flying in a horizontal circle, looking for lunch. When the vulture is headed directly at you, you hear its shrieks at a frequency f_1. When it is headed directly away, you hear its shrieks at a frequency f_2. It is not unreasonable to assume that all the shrieks are emitted exactly alike.

- 1a) (10 points) At what frequency will the vulture hear its shrieks?

$$\frac{f_{obs}}{f_{src}} = \frac{v - v_{obs}}{v - v_{src}}$$

$me \xrightarrow{+} vulture$

$$\cancel{\frac{f_1}{f_{src}} = \frac{v}{v - v_{src}}} \qquad \cancel{\frac{f_2}{f_{src}} = \frac{v}{v + v_{src}}}$$

observer: vulture

$$\frac{f_{obs}}{f_1} = \frac{v + v_{obs}}{v} \qquad \frac{f_{obs}}{f_2} = \frac{v - v_{obs}}{v}$$

$$\boxed{f_{obs} = f_1\left(\frac{v+v_{obs}}{v}\right) \qquad f_{obs} = f_2\left(\frac{v-v_{obs}}{v}\right)}$$

See number 1b for v_{obs}. (denoted as v_{src} there)

$f_{obs} = ?$ 8/10

- 1b) (10 points) How fast is the vulture moving?

$vulture \xrightarrow{+} me$

$$\frac{f_1}{f_{src}} = \frac{v}{v - v_{src}}$$

$$\frac{f_2}{f_{src}} = \frac{v}{v + v_{src}}$$

$$f_1(v - v_{src}) = f_2(v + v_{src})$$

$$f_1 v - f_1 v_{src} = f_2 v + f_2 v_{src}$$

$$v_{src}(f_1 + f_2) = v(f_1 - f_2)$$

$$\boxed{v_{src} = \frac{v(f_1 - f_2)}{f_1 + f_2}}$$

Source: vulture

10/10

src $\qquad\qquad$ + $\qquad\qquad$ obs

vulture $\longrightarrow$ me

- 1c) (10 points) You begin to walk away from the vulture with a speed V_x. What frequencies will you hear now, when the vulture is flying straight towards you (f_1') and straight away (f_2')? How does the ratio f_1'/f_2' compare to f_1/f_2?

$$\frac{f_1'}{f_{src}} = \frac{v + V_x}{v + V_{vul}} \qquad \frac{f_2'}{f_{src}} = \frac{v + V_x}{v - V_{vul}}$$

$$\boxed{f_1' = f_{src}\left(\frac{v+V_x}{v+V_{vul}}\right) \qquad f_2' = f_{src}\left(\frac{v+V_x}{v-V_{vul}}\right)}$$

Note: $f_{src} = \dfrac{f_1(v+V_{vul})}{v}$

V_{vul} is defined in 1b) as V_{src}.

$$\frac{f_1'}{f_2'} = \frac{v - V_{vul}}{v + V_{vul}}$$

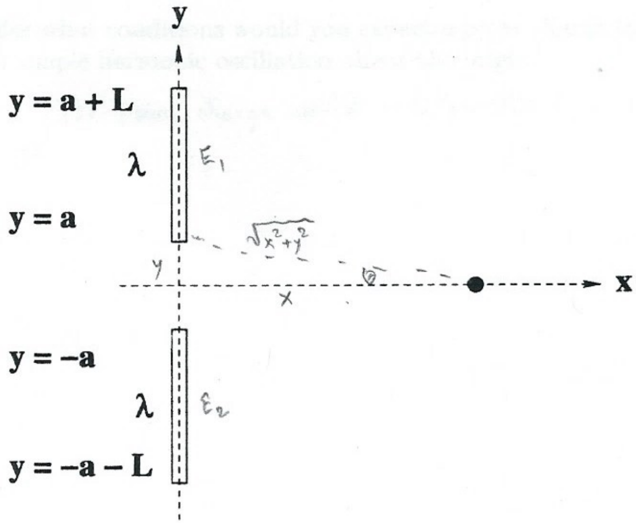
$$\frac{f_1}{f_{src}} = \frac{v}{v + V_{vul}} \qquad \frac{f_2}{f_{src}} = \frac{v}{v - V_{vul}}$$

$$\boxed{\frac{f_1}{f_2} = \frac{v - V_{vul}}{v + V_{vul}} = \frac{f_1'}{f_2'}}$$ ✓

$f_1' = ?$

$f_2' = ?$

8/10



2) Two identical non-conducting rods of length L and linear charge density λ are placed along the y-axis so that the origin lies at the center of a gap of width $2a$ that sits between them, as shown.

15

- 2a) (15 points) Find the (vector) electric field at all points on the positive x-axis.

$\boxed{E_y = 0 \text{ by symmetry}}$

$E_x = E_1 + E_2$

$E_1 = E_2$

$dE = 2 \cdot \frac{dq}{4\pi\epsilon_0 r^2} \cos\theta$

$= 2 \cdot \frac{dq}{4\pi\epsilon_0 (x^2+y^2)} \cdot \frac{x}{\sqrt{x^2+y^2}}$

$= 2 \cdot \frac{x\, dq}{4\pi\epsilon_0 (x^2+y^2)^{3/2}}$

$dq = \lambda\, dy$

$= \frac{\lambda}{L} dy$

$dE = \frac{\lambda x}{2\pi\epsilon_0} \cdot \frac{1}{(x^2+y^2)^{3/2}} dy$

$\int_a^{a+L} \frac{1}{(x^2+y^2)^{3/2}} dy$

$y^2 = x^2 \tan^2\theta$

$y = x \tan\theta$

$dy = x \sec^2\theta\, d\theta$

$\int_{\tan^{-1}(\frac{a}{x})}^{\tan^{-1}(\frac{a+L}{x})} \frac{x \sec^2\theta}{x^3 \sec^3\theta} d\theta$

$\int \frac{1}{x^2 \sec\theta} d\theta$

$\int \frac{1}{x^2} \cos\theta\, d\theta$

$\frac{1}{x^2} \sin\theta \Big]_{\tan^{-1}(\frac{a}{x})}^{\tan^{-1}(\frac{a+L}{x})} =$

$\frac{1}{x^2}\left(\frac{a+L}{\sqrt{(a+L)^2+x^2}} - \frac{a}{\sqrt{a^2+x^2}}\right)$

$\boxed{E = \frac{\lambda}{2\pi\epsilon_0 x}\left(\frac{a+L}{\sqrt{(a+L)^2+x^2}} - \frac{a}{\sqrt{a^2+x^2}}\right)\hat{i}}$

3

- 2b) (5 points) Under what conditions would you expect a point charge (q, m) released at rest on the $x - axis$, to undergo simple harmonic oscillation about the origin?

The point charge would undergo SHO if $x << a+L$ and $x << a$.

10

- 2c) (10 points) Suppose a point charge was released under the conditions you stated above. Show that it will indeed execute simple harmonic motion and find the angular frequency of that motion.

$x << a+L$
$x << a$

$$E = \frac{\lambda}{2\pi\epsilon_0 x}\left(\frac{a+L}{\sqrt{(a+L)^2+x^2}} - \frac{a}{\sqrt{a^2+x^2}}\right)$$

$$= \frac{\lambda}{2\pi\epsilon_0 x}\left(\frac{1}{\sqrt{1+\frac{x^2}{(a+L)^2}}} - \frac{a}{\sqrt{a^2+x^2}}\right)$$

$$= \frac{\lambda}{2\pi\epsilon_0 x}\left(\left(1 - \frac{x^2}{2(a+L)^2}\right) - \frac{1}{\sqrt{1+\frac{x^2}{a^2}}}\right)$$

$$= \frac{\lambda}{2\pi\epsilon_0 x}\left(1 - \frac{x^2}{2(a+L)^2} - 1 + \frac{x^2}{2a^2}\right)$$

$$= \frac{\lambda}{2\pi\epsilon_0 x}\left(\frac{x^2}{2a^2} - \frac{x^2}{2(a+L)^2}\right)$$

$$= \frac{\lambda x}{2\pi\epsilon_0}\left(\frac{1}{2a^2} - \frac{1}{2(a+L)^2}\right)$$

$\vec{F} = q\vec{E}$

$$m\ddot{x} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{2a^2} - \frac{1}{2(a+L)^2}\right)x$$

$$\omega_0^2 = \frac{\lambda}{2\pi\epsilon_0 m}\left(\frac{1}{2a^2} - \frac{1}{2(a+L)^2}\right)$$

$$\boxed{\omega_0 = \sqrt{\frac{\lambda |q|}{2\pi\epsilon_0 m}\left(\frac{1}{2a^2} - \frac{1}{2(a+L)^2}\right)}}$$

A very long cylindrical distribution of radius R has a volume charge density given by:

$$\rho(\vec{r}) = \frac{6\lambda_0}{\pi R^4}\, r\,(R-r)$$

where r is measured from the longitudinal symmetry axis of the distribution.

- 3a) (10 points) How much charge is contained in a coaxial cylinder of radius r and length x? Consider both the case where $r < R$ *and* $r > R$.

$r < R$

$$dq = \rho(\vec{r})\, dr$$

$$q = \int_0^r \frac{6\lambda_0}{\pi R^4} r(R-r)\, dr$$

$$= \int_0^r \frac{6\lambda_0 r}{\pi R^3} - \frac{6\lambda_0 r^2}{\pi R^4}\, dr$$

$$\boxed{Q = \frac{3\lambda_0 r^2}{\pi R^3} - \frac{2\lambda_0 r^3}{\pi R^4}}$$

$r > R$

$$q = \int_0^R \rho(\vec{r})\, dr$$

$$Q = \frac{3\lambda_0 R^2}{\pi R^3} - \frac{2\lambda_0 R^3}{\pi R^4}$$

$$= \frac{3\lambda_0}{\pi R} - \frac{2\lambda_0}{\pi R}$$

$$\boxed{Q = \frac{\lambda_0}{\pi R}}$$

9

- 3b) (10 points) Find the electric field at all points inside and outside the cylinder.

$r < R$

$$E(2\pi r x) = \frac{1}{\epsilon_0}\left(\frac{3\lambda_0 r^2}{\pi R^3} - \frac{2\lambda_0 r^3}{\pi R^4}\right)$$

$$\boxed{E = \frac{1}{2\pi r x \epsilon_0}\left(\frac{3\lambda_0 r^2}{\pi R^3} - \frac{2\lambda_0 r^3}{\pi R^4}\right)}$$

$r > R$

$$E(2\pi r x) = \frac{\lambda_0}{\pi R \epsilon_0}$$

$$\vec{E} = \frac{\lambda_0}{2\epsilon_0 R r x \pi^2}$$

$$= \boxed{\frac{\lambda_0}{2\pi^2 \epsilon_0 R r x}}$$

8

- 3c) (10 points) Use $r = R$ as a reference and find the potential for all points inside and outside the cylinder.

$r < R$

$$E = \frac{1}{2\pi r x \epsilon_0}\left(\frac{3\lambda_0 r^2}{\pi R^3} - \frac{2\lambda_0 r^3}{\pi R^4}\right)$$

$$= \frac{3\lambda_0 r}{2\pi^2 x R^3 \epsilon_0} - \frac{\lambda_0 r^2}{\pi^2 x R^4 \epsilon_0}$$

$$V = -\int_R^r E\, dr$$

$$\left.\frac{3\lambda_0 r^2}{4\pi^2 x R^3 \epsilon_0} - \frac{\lambda_0 r^3}{3\pi^2 x R^4 \epsilon_0}\right]_r^R$$

$$\boxed{V = \frac{3\lambda_0}{4\pi^2 x R \epsilon_0} - \frac{\lambda_0}{3\pi^2 x R \epsilon_0} - \frac{3\lambda_0 r^2}{4\pi^2 x R^3 \epsilon_0} + \frac{\lambda_0 r^3}{3\pi^2 R^4 \epsilon_0}}$$

2

$r > R$

$$V = -\int_R^r \frac{\lambda_0}{2\pi^2 \epsilon_0 R r x}\, dr$$

$$\frac{\lambda_0}{2\pi \epsilon_0 R x} \ln|r|$$

$$\boxed{V = \frac{\lambda_0}{2\pi \epsilon_0 R x} \ln\left(\frac{R}{r}\right)}$$

3

3