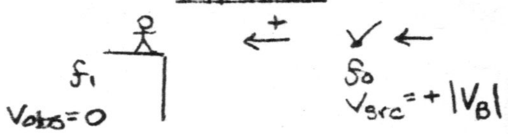
$$\frac{f_{obs}}{f_{src}} = \frac{V_{snd} - V_{obs}}{V_{snd} - V_{src}}$$

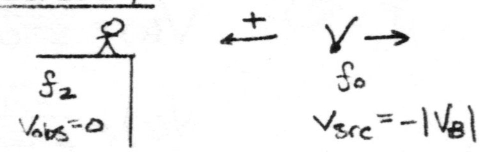
1) You're standing on the edge of a cliff. Out in the distance, at roughly the same elevation as you, you see a vulture flying in a horizontal circle, looking for lunch. When the vulture is headed directly at you, you hear its shrieks at a frequency f_1. When it is headed directly away, you hear its shrieks at a frequency f_2. It is not unreasonable to assume that all the shrieks are emitted exactly alike.

- 1a) (10 points) At what frequency will the vulture hear its shrieks?

Approach



Recession



$$\frac{f_1}{f_0} = \frac{V_{snd}}{V_{snd} - |V_B|} \qquad \frac{f_2}{f_0} = \frac{V_{snd}}{V_{snd} + |V_B|}$$

$$f_0\left(\frac{1}{f_1} + \frac{1}{f_2}\right) = 2$$

$$\boxed{f_0 = \frac{2 f_1 f_2}{f_1 + f_2}}$$

- 1b) (10 points) How fast is the vulture moving?

$$f_0\left(\frac{1}{f_2} - \frac{1}{f_1}\right) = \frac{2|V_B|}{V_{snd}}$$

$$\frac{2 f_1 f_2}{f_1 + f_2} \; \frac{f_1 - f_2}{f_1 f_2} = \frac{2|V_B|}{V_{snd}}$$

$$\boxed{|V_B| = V_{snd} \, \frac{f_1 - f_2}{f_1 + f_2}}$$

- 1c) (10 points) **You begin to walk away from the vulture with a speed V_x. What frequencies will you hear now, when the vulture is flying straight towards you (f_1') and straight away (f_2')? How does the ratio f_1'/f_2' compare to f_1/f_2?**

Now $V_{obs} \rightarrow +|V_x|$

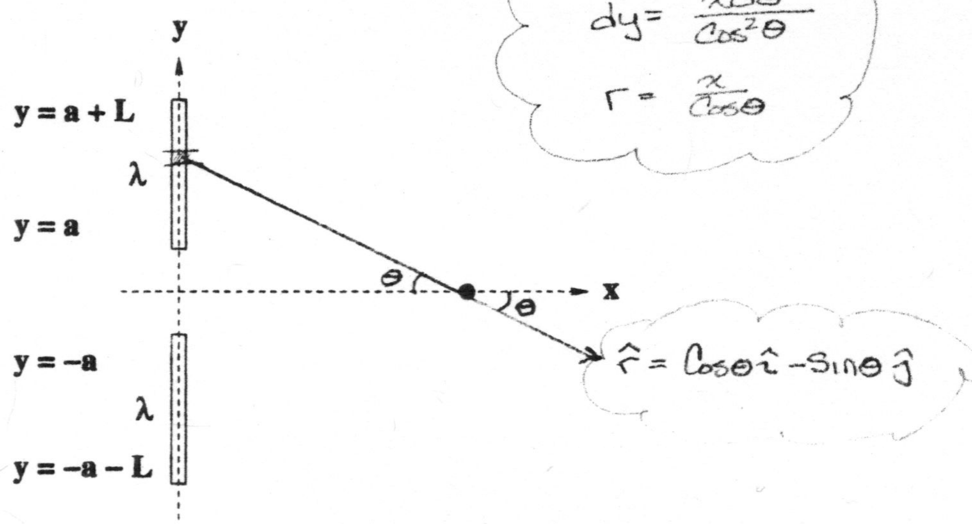
$$\frac{f_1'}{f_0} = \frac{V_{snd} - |V_x|}{V_{snd} - |V_B|} \qquad \frac{f_2'}{f_0} = \frac{V_{snd} - |V_x|}{V_{snd} + |V_B|}$$

Compare these equations to the original set up in part a:

$$\boxed{\begin{gathered}\frac{f_1'}{f_2'} = \frac{f_1}{f_2} \\ f_1' = f_1\left(1 - \frac{|V_x|}{V_{snd}}\right) \\ f_2' = f_2\left(1 - \frac{|V_x|}{V_{snd}}\right)\end{gathered}}$$

$dq = \lambda ds$
$dq = \lambda dy$

$y = x\tan\theta$
$dy = \dfrac{x\,d\theta}{\cos^2\theta}$
$r = \dfrac{x}{\cos\theta}$



2) Two identical non-conducting rods of length L and linear charge density λ are placed along the y-axis so that the origin lies at the center of a gap of width $2a$ that sits between them, as shown.

- 2a) (15 points) Find the (vector) electric field at all points on the positive x-axis.

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 r^2}\hat{r}$$

$$d\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\,\frac{x\,d\theta}{\cos^2\theta}\,\frac{\cos^2\theta}{x^2}\,(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$d\vec{E} = \frac{\lambda}{4\pi\epsilon_0 x}\,d\theta\,(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0 x}\,(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})\Big|_{\theta_i}^{\theta_f}$$

The first few lines are very general... but now we have to pay attention :)
Technically, λ is piecewise, so we'll introduce that as we evaluate the limits...

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0 x}\left[\left(\frac{a+L}{\sqrt{x^2+(a+L)^2}} - \frac{a}{\sqrt{x^2+a^2}} + \frac{(-a)}{\sqrt{x^2+a^2}} - \frac{-(a+L)}{\sqrt{x^2+(a+L)^2}}\right)\hat{\imath} + \left(\frac{x}{\sqrt{x^2+(a+L)^2}} - \frac{x}{\sqrt{x^2+a^2}} + \frac{x}{\sqrt{x^2+a^2}} - \frac{x}{\sqrt{x^2+(a+L)^2}}\right)\hat{\jmath}\right]$$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 x}\left(\frac{a+L}{\sqrt{x^2+(a+L)^2}} - \frac{a}{\sqrt{x^2+a^2}}\right)\hat{\imath}}$$

This is, of course, easier if you invoke symmetry up front - but where's the fun in that?!? :)

* Alternately, you can superpose the fields due to two finite lines at points along their bisector

- 2b) (5 points) **Under what conditions would you expect a point charge (q, m) released at rest on the $x-axis$, to undergo simple harmonic oscillation about the origin?**

- x should be small compared to a
- $q\lambda = -|q\lambda|$

- 2c) (10 points) **Suppose a point charge was released under the conditions you stated above. Show that it will indeed execute simple harmonic motion and find the angular frequency of that motion.**

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 x}\left[\left(1+\frac{x^2}{(a+L)^2}\right)^{-1/2} - \left(1+\frac{x^2}{a^2}\right)^{-1/2}\right]\hat{i}$$

$x \ll a$

$$\vec{E} \simeq \frac{\lambda}{2\pi\epsilon_0 x}\left[\frac{1}{2}\frac{x^2}{a^2} - \frac{1}{2}\frac{x^2}{(a+L)^2}\right]\hat{i}$$

$$\vec{E} \simeq \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{a^2} - \frac{1}{(a+L)^2}\right)x\,\hat{i}$$

Linear restoring force

The force on q: $\vec{F_E} = q\vec{E} = \frac{-|\lambda q|}{4\pi\epsilon_0}\left(\frac{1}{a^2} - \frac{1}{(a+L)^2}\right)x\,\hat{i}$

q experiences a linear restoring force $\Rightarrow$ SHO

$$\omega = \sqrt{\frac{|\lambda q|}{4\pi\epsilon_0 m}\left(\frac{1}{a^2} - \frac{1}{(a+L)^2}\right)}$$

if $\vec{F} = -k\vec{x}$ $\omega = \sqrt{k/m}$

A very long cylindrical distribution of radius R has a volume charge density given by:

$$\rho(\vec{r}) = \frac{6\lambda_0}{\pi R^4}\, r\,(R - r)$$

where r is measured from the longitudinal symmetry axis of the distribution.

- 3a) (10 points) How much charge is contained in a coaxial cylinder of radius r and length x? Consider both the case where $r < R$ *and* $r > R$.

Cylindrical symmetry: $dq = \rho\, dV = \rho\, 2\pi r\, dr\, x$

$(r<R)$ $q_{in}(r,x) = \int dq = \frac{6\lambda_0}{\pi R^4} 2\pi x \int_0^r (Rr^2 - r^3)\, dr = 12\lambda_0 x \left(\frac{1}{3}\frac{r^3}{R^3} - \frac{1}{4}\frac{r^4}{R^4}\right)$

$q_{in}(r,x) = \lambda_0 x \left(4\frac{r^3}{R^3} - 3\frac{r^4}{R^4}\right)$

$(r>R)$ $q_{in}(r,x) = q_{in}(R,x) = \lambda_0 x$

$$\boxed{q_{in}(r,x) = \begin{cases} \lambda_0 x \frac{r}{R}\left(4\frac{r^2}{R^2} - 3\frac{r^3}{R^3}\right) & (r \le R) \\ \lambda_0 x & (r \ge R) \end{cases}}$$

- 3b) (10 points) Find the electric field at all points inside and outside the cylinder.

Cyl. Symmetry: $\vec{E} = \frac{\lambda_{in}(r)}{2\pi\epsilon_0 r}\hat{r} = \frac{q_{in}(r,x)/x}{2\pi\epsilon_0 r}\hat{r}$

$$\boxed{\vec{E} = \begin{cases} \frac{\lambda_0}{2\pi\epsilon_0 R}\left(4\frac{r^2}{R^2} - 3\frac{r^3}{R^3}\right)\hat{r} & (r \le R) \\ \frac{\lambda_0}{2\pi\epsilon_0 r}\hat{r} & (r \ge R) \end{cases}}$$

- 3c) (10 points) **Use $r = R$ as a reference and find the potential for all points inside and outside the cylinder.**

$$V(r) = -\int_R^r E(r)\,dr \qquad \text{(Cyl Sym)}$$

$(r \le R)$

$$V(r) = \frac{-\lambda_0}{2\pi\epsilon_0 R}\int_R^r \left(4\frac{r^2}{R^2} - 3\frac{r^3}{R^3}\right) dr$$

$$V(r) = \frac{-\lambda_0}{2\pi\epsilon_0 R}\left(\frac{4}{3}\frac{r^3}{R^2} - \frac{3}{4}\frac{r^4}{R^3}\right)\Bigg|_R^r$$

$$V(r) = \frac{-\lambda_0}{2\pi\epsilon_0}\left(\frac{4}{3}\frac{r^3}{R^3} - \frac{3}{4}\frac{r^4}{R^4} - \frac{4}{3} + \frac{3}{4}\right)$$

$$V(r) = \frac{-\lambda_0}{24\pi\epsilon_0}\left(16\frac{r^3}{R^3} - 9\frac{r^4}{R^4} - 7\right)$$

$(r \ge R)$

$$V(r) = \frac{-\lambda_0}{2\pi\epsilon_0}\int_R^r \frac{dr}{r}$$

$$V(r) = -\frac{\lambda_0}{2\pi\epsilon_0}\ln\left(\frac{r}{R}\right)$$

$$\boxed{V(r) = \begin{cases} \dfrac{\lambda_0}{24\pi\epsilon_0}\left(7 + \dfrac{9r^4}{R^4} - 16\dfrac{r^3}{R^3}\right) & (r \le R) \\[2ex] -\dfrac{\lambda_0}{2\pi\epsilon_0}\ln\left(\dfrac{r}{R}\right) & (r \ge R) \end{cases}}$$