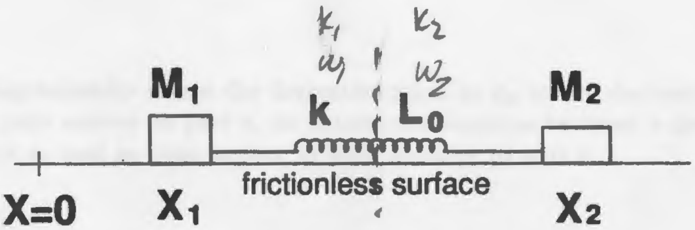
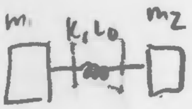
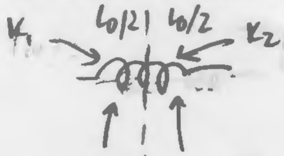


1) A pair of masses, M_1 and M_2 are joined by a spring of constant k and natural (unstretched) length L_0.

+0

- 1a) (5 points) Find the amount the spring is extended or compressed (Δx) as a function of the position of each mass (x_1 and x_2, respectively). Make sure Δx is positive when the spring is extended and negative when it's compressed.





$F = -kx$

$x(t) = A\cos(\omega t + \phi)$

$F = -kx$

$$\boxed{\Delta x = \frac{k \cdot M_1}{L_0} + \frac{k \cdot M_2}{L_0}}$$

$\omega_1 = \sqrt{\frac{k_1}{m_1}}$ $\omega_2 = \sqrt{\frac{k_2}{m_2}}$ $\omega = \sqrt{\frac{k}{m}}$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_1 + k_2}{k_1 k_2} \Rightarrow k = \frac{k_1 k_2}{k_1 + k_2}$$

+1

- 1b) (5 points) Use Newton's laws to obtain differential equations (written in terms of x_1 and x_2) that describe the motion of each mass. Note that these are 'coupled' equations - they will each depend on both x_1 and x_2 - but not to worry, we'll address that later. Make sure the derivative term in each equation has the correct sign when the spring is extended and compressed!

$$F_{net} \Rightarrow -kx \Rightarrow ma$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}(\Delta x) = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x(x_1, x_2) = 0$$

$$= \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}\left(\frac{kM_1}{L_0} + \frac{kM_2}{L_0}\right) = 0}$$

- 1c) (5 points) Algebraically relate the derivative term in x_1 to the derivative term in x_2. Use this result, along with your answer to part *a*, to obtain relationships between a derivative of Δx and each of the derivatives of x_1 and x_2 that appear in your answers to part *b*.

- 1d) (10 points) Now it's time to put it all together - rewrite the differential equations from part *b* in terms of Δx. The results should look familiar. Find the solution for Δx as a function of time (make sure you evaluate, in terms of given information, any constants that are determined by the construction of the system).

- 1e) (5 points) Suppose we were to tie one end of the spring off to a wall, and the other end of the spring to a mass M. What value would M have to have in order for this new system to oscillate with the same period as the two-mass system we've been working on? This value, known as the *reduced mass*, is used to simplify the discussion of binary systems (diatomic molecules, for instance) in oscillation.

2) Recall that the amplitude of a driven mass-spring system is given by

$$A(\Omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (\frac{b\Omega}{m})^2}}$$

- **2a) (10 points)** Show that the following relationship holds true for a mass-spring system:

$$\omega_{damp}^2 = \frac{1}{2}\left(\Omega_{res}^2 + \omega_0^2\right)$$

For a driven system:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos(\Omega t) \qquad \text{if } \omega_0 \gg \frac{b}{2m}$$

$$\omega_{damp} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \omega_0^2\sqrt{1-\left(\frac{b}{2m\omega_0}\right)^2} \approx \omega_0\left(1 - \frac{1}{2}\left(\frac{b}{2m\omega_0}\right)\right) = \omega_0$$

$$\Omega_{res} = \omega_0\sqrt{1-2\left(\frac{b}{2m\omega_0}\right)^2} \approx \omega_0\left(1-\frac{1}{2}\left(\frac{b}{2m\omega_0}\right)\right) \approx \omega_0$$

if $\omega_0 \gg \frac{b}{2m}$

+6

Hence,

$$\omega_{damp}^2 = \omega_0^2 = \frac{1}{2}\left(\Omega_{res}^2 + \omega_0^2\right) = \frac{1}{2}\left(\omega_0^2 + \omega_0^2\right)$$

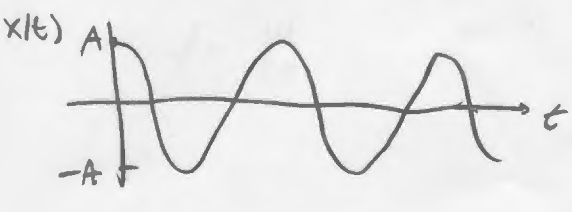
$$\omega_0^2 = \frac{1}{2}(2\omega_0^2) \Rightarrow \boxed{\omega_0^2 = \omega_0^2}$$

- **2b) (5 points)** Convince the grader that the relationship in part a holds for all simple harmonic oscillators.

For simple harmonic oscillators,

$\omega_{damp}^2 = \omega_0^2$, since no damping force exists.

Since the SHO is constant in amplitude and periodic, as per the displacement graph below:



,

$\Omega_{res} = \omega_0$.

+3

Hence, $\omega_{damp}^2 = \frac{1}{2}\left(\Omega_{res}^2 + \omega_0^2\right)$ simplifies to

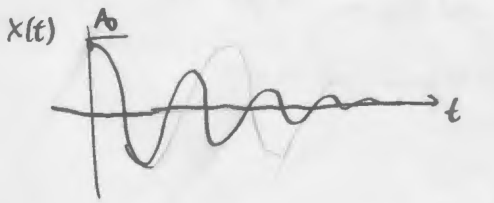
$$\omega_0^2 = \frac{1}{2}(2\omega_0^2) = \omega_0^2$$

$$\boxed{\omega_0^2 = \omega_0^2}$$

- 2c) (10 points) On the way to class, you spot an abandoned bird's nest sitting in a low-hanging, more-or-less horizontal branch. You displace the end by some small amount and release it, and note that *a*) the tip makes about f_1 complete cycles every second, and *b*) it takes about N complete cycles for the amplitude of the branch's vibrations to drop to half its initial value. Find the natural frequency (f_0) for the branch/nest system. [*Careful.* These are f's, not ω's. f's are easier to observe directly.]

$x(t) = A_0 e^{-\frac{bt}{2m}} \cos(\omega t + \phi)$

$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$



$T = \frac{1}{f_1}$

$\omega = 2\pi f_1 = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$

$\omega_0^2 - \left(\frac{b}{2m}\right)^2 = 2\pi f_1$

$\omega_0^2 = 4\pi^2 f_1^2 + \left(\frac{b}{2m}\right)^2$

$A(t) = A_0 e^{-\frac{bt}{2m}}$

At some t_N,

$A(t_N) = \frac{A_0}{2} = A_0 e^{-\frac{bt}{2m}} = A_0 e^{-\frac{bNT}{2m}}$

$\frac{1}{2} = e^{-\frac{bNT}{2m}}$

$\cancel{-}\ln(2) = \cancel{-}\frac{bNT}{2m}$

$\frac{2m(\ln(2))}{b} = \frac{N}{f_1}$

$N = \frac{2m(\ln(2))}{b} f_1$

$\omega_0 = \sqrt{4\pi^2 f_1^2 + \frac{b^2}{4m^2}}$

$f_1 = \frac{Nb}{2m(\ln(2))}$

$\boxed{\omega_0 = \frac{b}{2m}\sqrt{\left(\frac{2\pi N}{\ln(2)}\right)^2 + 1}}$

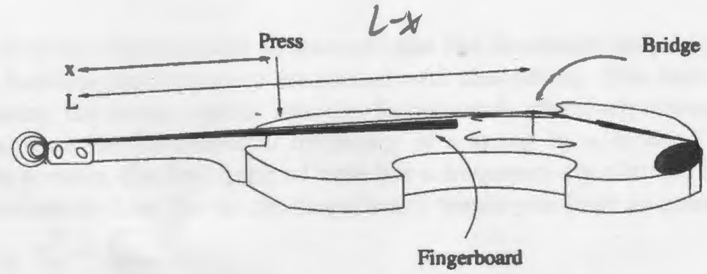
$\boxed{f_0 = \frac{b}{4\pi m}\sqrt{\left(\frac{2\pi N}{\ln(2)}\right)^2 + 1}}$

+10

- 2d) (5 points) Suppose we want to knock that old nest out of the tree. What would be the most effective frequency to shake the branch at?

$W = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$

$f = \frac{\omega}{2\pi}$



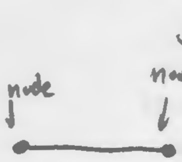
- **3a) (5 points) (Carefully) pluck a string on a violin. For a brief moment, the pluck generates noise - but that noise quickly gives way to music. Explain, in terms of physics, what is happening.**

In terms of physics, the initial pluck (before released) of the string gives it elastic potential energy, which is then converted to kinetic energy when the string is released. This kinetic energy plays out in many frequencies initially being heard — the "noise". However, due to the boundary conditions of the string (nodes on the ends), many frequencies are dampened out and only resonant frequencies — the harmonics to the fundamental frequency — are heard, hence giving the musical sound. The power spectrum plays a role in the dominant harmonic heard, but the overall sound becomes musical.

+3

- **3b) (5 points) *Derive* the set of resonance frequencies for one of the open (that is, un-fingered) strings in terms of its effective length (L), the volume mass density of the material it is made of (ρ), the diameter of the string (D) and the tension in the string (T).**

$f = \frac{v_x}{2L}(N)$



$$\Rightarrow v_x = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho\pi\frac{D^2}{4}}} = \sqrt{\frac{4T}{\rho\pi D^2}} = 2\sqrt{\frac{T}{\rho\pi D^2}}$$

$\rho = \frac{g}{cm^3}$

Volume $= \pi\left(\frac{D}{2}\right)^2 L$

Set of frequencies: $$\boxed{\frac{2\sqrt{\frac{T}{\rho\pi D^2}}}{2L}\cdot N = \frac{\sqrt{\frac{T}{\rho\pi D^2}}}{L}\cdot N}$$

+5

- 3c) (5 points) It is not unreasonable to assume that the dominant frequency heard from an excited string will be the fundamental frequency associated with that string. One may change the fundamental frequency by pressing the string tightly into the fingerboard, effectively changing its length. Suppose you wanted to increase the fundamental frequency of a string by a factor F. Where (x) would you have to press? On a violin, the first fingered note has a frequency equal to 1.12 times the open-stringed frequency. Approximately how far up the fingerboard would you have to press to generate it?

$f = \frac{V_x}{2L}(N)$ $N=1$, $f_0 = \frac{V_x}{2L}$

$\hookrightarrow f_0 \cdot F = \frac{V_x}{2L}\cdot F \Rightarrow \frac{\cancel{V_x}}{\cancel{2}(x)} = \frac{\cancel{V_x}}{\cancel{2}L}\cdot F$

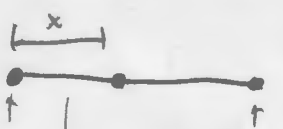
$\frac{1}{x} = \frac{F}{L}$

$\boxed{x = \frac{L}{F}}$

on the violin, need to go up a distance of $\frac{L}{1.12}$ up the fingerboard to generate it.

+3

- 3d) (10 points) Another way to change the dominant frequency you hear is to **lightly** press the string in some magic spot. Since the string is not tightly pressed, it is still able to vibrate on either side of the finger - this has the effect of imposing an intermediate node on the system, emphasizing some harmonic over the fundamental. Find each of these magic spots (x) and the frequency associated with it (in terms of the fundamental).



new fundamental f_0 has $\frac{\lambda}{2} = \frac{x}{L}$

$V_x = f\lambda = f\left(\frac{x}{L}\right)N \Rightarrow f = \frac{V_x \cdot L}{x}\cdot N$

$f = \frac{V_x}{2L}\cdot N$

$\boxed{= 2\sqrt{\frac{T}{\pi D^2}}\cdot\frac{LN}{x} \text{ for each } x}$

+2

- 3e) (5 points) Bowing near the bridge can make a much nicer sound than bowing off towards the fingerboard. Explain in terms of physics why this is so.

Bowing creates an intermediate node in the string. Bowing near the bridge effectively creates that intermediate node closer to the actual end of the string, allowing for the resonant frequencies of the string to be closer to the nicer, musical resonant frequencies of the string with relationship $f = \frac{\sqrt{\frac{T}{\pi D^2}}\cdot N}{L}$.

+5