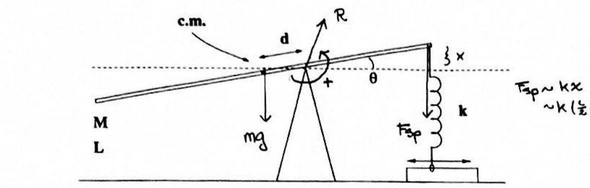


A uniform stick of mass m and length L is pivoted a distance d from its center-of-mass. A spring of constant k is located a distance $L/2$ from the center-of-mass, on the opposite side of the pivot (as shown). The spring is unstretched when the stick is horizontal. To make things easy, the bottom of the spring is connected to ring that is free to slide over a horizontal rail so that the spring is always oriented vertically.

- **1a) (10 points)** Find each of the torques (relative to the pivot) that act on the stick when it is displaced an amount θ, as shown. Do not assume small angles. For full credit, take out-of-the-page (counter-clockwise) as the positive rotational direction and make sure that the sign of each torque is consistent with the sign of the displacement.

$|\vec{\tau}| = r_\perp F$

take ccw (out of page) as + direction

$$\boxed{\begin{aligned} \tau_g &= mgd\cos\theta \\ \tau_R &= 0 \\ \tau_{sp} &= -k\left(\tfrac{L}{2}-d\right)^2\cos\theta\sin\theta \end{aligned}}$$

- **1b) (10 points)** Write the differential equation that describes the equation of motion for the stick. Do not assume small angles. Hint: For a uniform stick, $I_{cm} = \frac{1}{12}ML^2$.

(N2) $\quad \Sigma\vec{\tau} = I\vec{\alpha} \qquad I = I_{cm} + md^2 = \frac{1}{12}mL^2 + md^2$

$$mgd\cos\theta - k\left(\tfrac{L}{2}-d\right)^2\cos\theta\sin\theta = \left(\tfrac{1}{12}mL^2 + md^2\right)\frac{d^2\theta}{dt^2}$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{k}{m}\frac{\left(\frac{L}{2}-d\right)^2}{\frac{1}{12}L^2+d^2}\cos\theta\sin\theta - \frac{gd}{\frac{1}{12}L^2+d^2}\cos\theta = 0}$$

- **1c) (5 points)** Now write the differential equation of motion in the limit of small angular displacements.

$\sin\theta \simeq \theta \qquad \cos\theta \simeq 1$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{k}{m}\frac{\left(\frac{L}{2}-d\right)^2}{\frac{1}{12}L^2+d^2}\theta \simeq \frac{gd}{\frac{1}{12}L^2+d^2}}$$

- **1d) (5 points)** Find the angular frequency that the system will oscillate at (in the limit of small angles) and the value of θ when the system is in equilibrium.

$$\boxed{\begin{aligned} \omega_0 &= \sqrt{\frac{k}{m}\frac{\left(\frac{L}{2}-d\right)^2}{\frac{1}{12}L^2+d^2}} \\ \theta_{eq} &= \frac{mgd}{k\left(\frac{L}{2}-d\right)^2} \end{aligned}}$$

remember: in equilibrium, $\frac{d^2\theta}{dt^2} = 0$

$\frac{d^2\theta}{dt^2} + \omega_0^2\theta = C$

$0 + \omega_0^2\theta_{eq} = C$

$\theta_{eq} = C/\omega_0^2$

2) Recall that the amplitude of a driven mass-spring system is given by

$$A(\Omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\Omega^2)^2 + \left(\frac{b\Omega}{m}\right)^2}}$$

- **2a) (10 points)** Show that the following relationship holds true for a mass-spring system:

$$\omega_{damp}^2 = \frac{1}{2}\left(\Omega_{res}^2 + \omega_0^2\right)$$

$\omega_{damp} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$

$\Omega_{res} = \sqrt{\omega_0^2 - 2\left(\frac{b}{2m}\right)^2}$

$\Omega_{res}^2 - 2\omega_{damp}^2 = -\omega_0^2$

$$\boxed{\omega_{damp}^2 = \frac{1}{2}\left(\Omega_{res}^2 + \omega_0^2\right)} \quad ✓$$

- **2b) (5 points)** Convince the grader that the relationship in part *a* holds for all simple harmonic oscillators.

For the mass, spring system: $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos(\Omega t)$

the combinations $\frac{b}{m}, \frac{k}{m}, \frac{F_0}{m}$ are just place-holders for positive coefficients... we could just as well write

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = \gamma\cos(\Omega t)$$

where $\beta, \omega_0, \gamma, ...$ are all related to the particulars of the system under consideration...

We merely used algebra to eliminate some placeholders.

$f = \frac{1}{T} = \frac{\omega}{2\pi}$

- **2c) (10 points)** On the way to class, you spot an abandoned bird's nest sitting in a low-hanging, more-or-less horizontal branch. You displace the end by some small amount and release it, and note that *a*) the tip makes about f_1 complete cycles every second, and *b*) it takes about N complete cycles for the amplitude of the branch's vibrations to drop to half its initial value. Find the natural frequency (f_0) for the branch/nest system. [*Careful.* These are f's, not ω's. f's are easier to observe directly.]

$f_1 = \frac{1}{T_1} = \frac{\omega_{damp}}{2\pi}$

$A(t) = A_0 e^{-\frac{bt}{2m}}$ ← There are those place-holders :)

$\frac{A_1}{2} = A_0 e^{-\frac{b(t_1+NT)}{2m}}$

$A_1 = A_0 e^{-\frac{bt_1}{2m}}$

$\ln(2) = \frac{bNT_1}{2m}$

$\frac{b}{2m} = \frac{\ln(2)}{NT_1}$

$\frac{b}{2m} = \frac{f_1\ln(2)}{N}$

$\omega_{damp}^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2$

$\omega_0^2 = \omega_{damp}^2 + \left(\frac{f_1\ln(2)}{N}\right)^2$

$(2\pi f_0)^2 = (2\pi f_1)^2 + \left(\frac{f_1}{N}\ln(2)\right)^2$

$f_0^2 = f_1^2 + f_1^2\left(\frac{\ln(2)}{2\pi N}\right)^2$

$$\boxed{f_0 = f_1\sqrt{1 + \left(\frac{\ln(2)}{2\pi N}\right)^2}}$$

- **2d) (5 points)** Suppose we want to knock that old nest out of the tree. What would be the most effective frequency to shake the branch at?

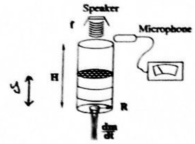
$2f_{damp}^2 = f_{res}^2 + f_0^2$ ← divide the relationship from (a) by $\left(\frac{1}{2\pi}\right)^2$

$f_{res}^2 = 2f_{damp}^2 - f_0^2$

$f_{res}^2 = 2f_1^2 - f_1^2\left[1 + \left(\frac{\ln(2)}{2\pi N}\right)^2\right]$

$f_{res}^2 = f_1^2\left[1 - \left(\frac{\ln(2)}{2\pi N}\right)^2\right]$

$$\boxed{f_{res} = f_1\sqrt{1 - \left(\frac{\ln(2)}{2\pi N}\right)^2}}$$



Mixed BC

$$f_{und} = \frac{V_{snd}}{4L}$$

$$f_N = (2N+1) f_{und}$$

Consider the apparatus shown above... Sound (of frequency f) is emitted by a speaker into a tube of radius R and height H. A microphone, placed near the open end of the tube, is used to monitor the intensity of the sound that is re-radiated from the tube. There is liquid (of volume mass density ρ) partially filling the tube, and it is leaking from a hole (of negligible area) in the bottom of the tube at a rate $\frac{dm}{dt}$ (m is mass).

- 3a) (10 points) What is the lowest frequency that will resonate in the tube? What are the boundary conditions at resonance? The more (correct) details you can give, the more points you will get.

Mixed boundary conditions
Node on top, Antinode on bottom (pressure)
Antinode on top, node on bottom (displacement)

$f_{und} = \frac{V_{snd}}{4L}$ and $L_{max} = H$, So...

$$\boxed{f_{und,\,low} = \frac{V_{snd}}{4H}}$$

- 3b) (10 points) Assuming the speed of sound in air is v_{snd}, how frequently will the microphone record intensity maxima as the water leaks out? [Hints: Under what condition will the re-radiated sound be at maximum intensity? How frequently does this condition occur? Call this frequency f_{peak} to avoid confusion with f].

$$f_N = (2N+1)\frac{V_{snd}}{4(H-y)}$$

← this is the usual way of writing things, but now f is fixed and y varies...

$$f = (2N+1)\frac{V_{snd}}{4(H-y_N)}$$

$$H - y_N = (2N+1)\frac{V_{snd}}{f}$$

$$2N+1 = (H-y_N)\, f / V_{snd}$$

$$2\frac{dN}{dt} = -\frac{f}{V_{snd}}\frac{dy_N}{dt}$$

$$m = \rho \pi R^2 y$$

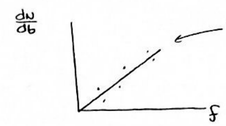
$$\frac{dm}{dt} = \rho \pi R^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\rho \pi R^2}\frac{dm}{dt}$$

$$\boxed{\frac{dN}{dt} = -\frac{f}{2\pi\rho R^2 V_{snd}}\frac{dm}{dt}}$$

- 3c) (10 points) If you're creative, you can use a device like this to measure the speed of sound in air. What parameter would you vary? What parameter would you record? Make a qualitative plot of one parameter vs. the other - and find v_{snd} as a function of the properties of that plot (for instance, if the plot is linear, How would you obtain v_{snd} from the measured slope and intercept?)

I would probably vary f and record $\frac{dN}{dt}$



Slope = $\frac{dm/dt}{2\pi\rho R^2 V_{snd}}$

$$\boxed{V_{snd} = \frac{dm/dt}{2\pi R^2 \rho \cdot \text{Slope}}}$$