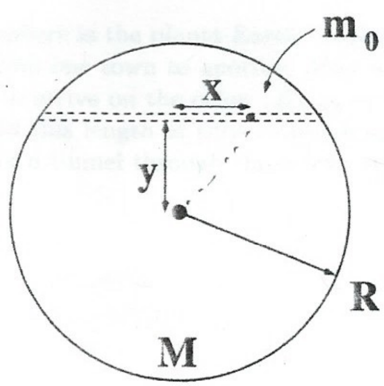


Inside a spherically-symmetric distribution of gravitational charge (mass), at at distance r from the center of the distribution, the gravitational field has a magnitude $g(r) = \frac{G\,m(r)}{r^2}$, where $m(r)$ is the amount of charge (mass) contained in a concentric sphere of radius r. The field itself points radially-inward, towards the center of the distribution.

5

- 1a) (5 points) Consider a uniform spherical distribution of mass M, radius R. Find $m(r)$ and $g(r)$ for all $r \leq R$.

$$m(R) = M$$

$$m(r) = \frac{\pi r^3}{\pi R^3} M = \frac{r^3}{R^3} M$$

$$g(r) = \frac{G}{r^2} \cdot \frac{r^3}{R^3} M = \frac{GMr}{R^3}$$

3

- 1b) (15 points) Now suppose a thin tunnel is drilled through the sphere, such that, at closest approach, an object in the tunnel would be a distance y from the center of the distribution (as shown). Show that if a point-charge (mass) m_0 is placed in the tunnel a distance x from that point of closest approach and released, it will execute simple-harmonic motion about the point of closest approach. Find the angular frequency associated with that motion.

$$r = \sqrt{x^2+y^2}$$

$$m\ddot{x} = -\frac{Gm(x^2+y^2)}{R^3}$$

$$\ddot{x} + \frac{G}{R^3}x^2 = -\frac{G}{R^3}y^2$$

- 1c) (10 points) Suppose the sphere is the planet Earth (neglect the fact that it is rotating) and the tunnel is used to deliver mail from one town to another. How long will it take for mail deposited (at rest) on one side of the tunnel to arrive on the other ($R_{earth} \approx 6.4 \times 10^6$ m, and - no - you don't get to use a calculator)? How would this length of time compare to the time required for mail, similarly deposited, to travel the length of a tunnel through the center of the Earth? Explain.

The motion of a driven oscillator is described by the differential equation

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 \, x = a \cos(\Omega t)$$

- 2a) (10 points) Write the equation of motion ($x(t)$) in as much detail as you can. The more correct detail you supply, the more points you will receive.

$$\frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$x(t) = A\cos(\omega t + \phi) + B\cos(\Omega t + \zeta)$

$A = e^{-\beta t}$

$\omega = \sqrt{\omega_0^2 - \beta^2}$

$B = \dfrac{a}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \left(\frac{b\Omega}{m}\right)^2}}$

$\tan \zeta = \dfrac{-\frac{b\Omega}{m}}{\omega_0^2 - \Omega^2}$

$m\ddot{x} = -kx - b\dot{x}$

$m\ddot{x} + b\dot{x} + kx = 0$

$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

$\frac{b}{m} = 2\beta$

$\beta = \frac{b}{2m}$

9/10

- 2b) (5 points) What angular frequency will the system oscillate at if $a = 0$?

If $a = 0$, the system will oscillate at ω, i.e. $\sqrt{\omega_0^2 - \beta^2}$

5/5

- 2c) (5 points) What (angular) driving frequency will maximize the amplitude of the system after it has reached steady-state?

$$\frac{d}{dt}\ (w_0^2-\Omega^2)^2+\left(\frac{b\Omega}{m}\right)^2=0$$

$$2(w_0^2-\Omega^2)(-2\Omega)+\frac{2b\Omega}{m}\left(\frac{b}{m}\right)=0$$

$$2(w_0^2-\Omega^2)-\frac{b^2}{m^2}=0$$

$$2w_0^2-2\Omega^2-\frac{b^2}{m^2}=0$$

$$\Omega^2=w_0^2-\frac{b^2}{2m^2}$$

$$\Omega=\sqrt{w_0^2-\frac{b^2}{2m^2}}$$

✓

5/5

- 2d) (10 points) Find the maximum amplitude attainable by the system in steady-state and show that it has a simple relationship to the quantity you calculated in part *b*.

$$B=\frac{\frac{a}{m}}{\sqrt{(w_0^2-\Omega^2)^2+\left(\frac{b\Omega}{m}\right)^2}}\qquad \Omega^2=w_0^2-\frac{b^2}{2m^2}$$

$$=\frac{\frac{a}{m}}{\sqrt{\left(w_0^2-w_0^2+\frac{b^2}{2m^2}\right)^2+\frac{b^2}{m^2}\left(w_0^2-\frac{b^2}{2m^2}\right)}}\qquad w=\sqrt{w_0^2-\beta^2}=\sqrt{w_0^2-\frac{b^2}{4m^2}}$$

$$=\frac{\frac{a}{m}}{\sqrt{\frac{b^4}{4m^4}+w_0^2\frac{b^2}{m^2}-\frac{b^4}{2m^4}}}$$

$$=\frac{\frac{a}{m}}{\sqrt{\frac{b^2}{m^2}\left(w_0^2+\frac{b^2}{2m^2}\right)}}$$

$$=\frac{a}{b\sqrt{w_0^2+\frac{b^2}{2m^2}}}$$

$$=\frac{a}{b\sqrt{w_0^2+\frac{1}{2}\beta^2}}$$

$$=\frac{a}{b\sqrt{w+\frac{3}{2}\beta^2}}$$

8/10

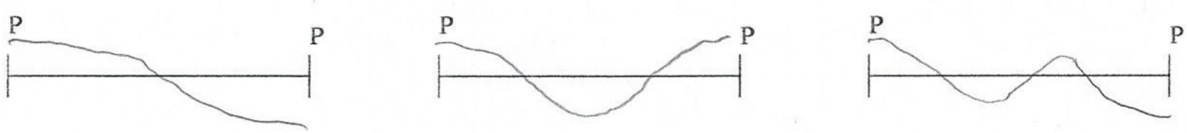
3) Consider a champagne glass driven at resonance...

- 3a) (5 points) Use the concept of phase to describe the boundary condition for resonant waves on the rim of the glass.

$\frac{n\lambda}{4} = \phi$?

0

- 3b) (15 points) Use the templates below to sketch the first three resonant modes on the rim of a champagne glass (hint: it might be easier if you sketch the sine waves on the horizontal lines first). In each case, describe how the motion would look to an external observer.



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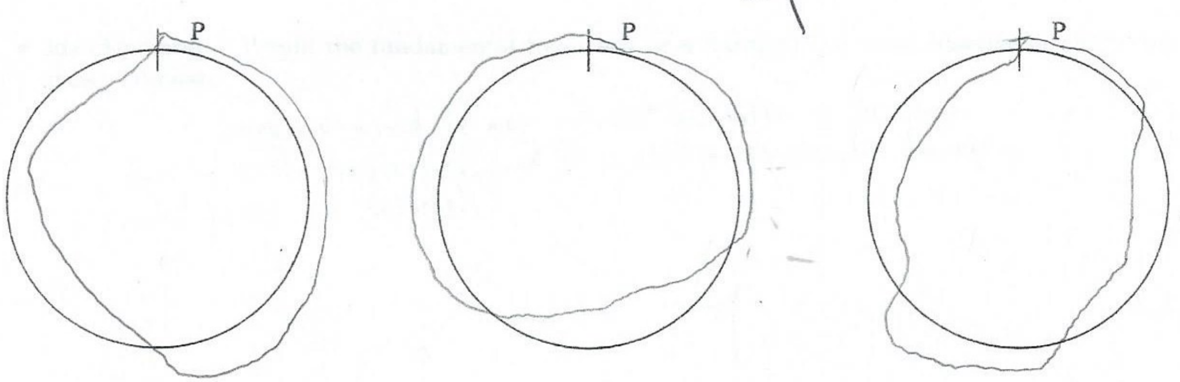

Look like caving in on one side.

Look like the first mode.

Oscillations with caving in on two sides.

- 3c) (5 points) If sound moves through the glass at a rate V_{snd} and the glass has a diameter D, what frequencies will cause the glass to vibrate with large amplitude displacements?

$C = \pi D$ $\quad\quad V_{snd} = \lambda f$

$\frac{n\lambda}{4} = \pi D$

$\frac{n V_{snd}}{4f} = \pi D$

$f = \frac{n V_{snd}}{4\pi D}$

3

- 3d) (5 points) Would the fundamental frequency or a harmonic be more effective at shattering the glass? Discuss.

A harmonic would be more efficient at shattering the glass due to higher frequencies, and thus shorter wavelengths (meaning more peaks and troughs).

4